REND. SEM. MAT. UNIV. PADOVA, Vol. 114 (2005)

$(S_3,\ S_6)$-Amalgams IV.

WOLFGANG LEMPKEN (*) - CHRISTOPHER PARKER (**)
PETER ROWLEY (***)

Introduction.

This paper in the fourth in a series of seven papers devoted to the study of $(S_3,\ S_6)$-amalgams. We continue the section numbering of the first three parts [LPR1], and refer the reader to Sections 1 and 2 for notation and background material. However for the readers' convenience we summarize some of the main points from these sections.

Adopting the philosophy of Goldschmidt [Go] our study of $(S_3,\ S_6)$-amalgams proceeds by examining the action of a group G (which is a certain free amalgamated product) on a certain tree Γ. Now G has two orbits on $V(\Gamma)$, the vertices of Γ, and one orbit on the edges of Γ. Let $a_1, a_2, \in V(\Gamma)$ be adjacent vertices. Then the properties of $(S_3,\ S_6)$-amalgams translate into this situation as follows:

1) $G = \langle G_{a_1}, G_{a_2} \rangle$;
2) $G_{a_1} \cap G_{a_2} = G_{a_1 a_2}$ contains no non-trivial normal subgroup of G;
3) $G_{a_1 a_2} \in Syl_2(G_{a_1}) \cap Syl_2(G_{a_2})$;
4) $C_{G_{a_i}}(O_2(G_{a_i})) \leq O_2(G_{a_i})$ for $i = 1, 2$; and
5) $G_{a_1}/O_2(G_{a_1}) \cong S_3$ and $G_{a_2}/O_2(G_{a_2}) \cong S_6$.

The overall aim is to determine the group theoretic structure of the vertex stabilizers G_{a_1} and G_{a_2}.

For $\delta \in V(\Gamma)$, we set

$$\Delta(\delta) = \{\lambda \in V(\Gamma) \mid d(\delta, \lambda) = 1\},$$

where $d(,)$ in the standard graph theoretic distance on Γ. Also for $i \in \mathbb{N}$,

(*) Indirizzo dell'A.: Institute for Experimental Mathematics, University of Essen, Ellernstrasse 29, Essen, Germany.

(**) Indirizzo dell'A.: School of Mathematics, University of Birmingham, Edgbaston, Birmingham B15 2TT, United Kingdon.

(***) Indirizzo dell'A.: School of Mathematics, The University of Manchester P.O. Box 88, Manchester M60 1QD, United Kingdom.

we let

$$\Delta^{[i]}(\delta) = \{\lambda \in V(\Gamma) \mid d(\delta, \lambda) \leq i\}.$$

The two orbits of G on $V(\Gamma)$ are, in fact, a_1^G and a_2^G; for $\delta \in V(\Gamma)$ we use $\delta \in O(S_3)$ (respectively $\delta \in O(S_6)$) to mean that $\delta \in a_1^G$ (respectively $\delta \in a_2^G$).

For $\delta \in V(\Gamma)$ put $Q_\delta = O_2(G_\delta)$. Various normal subgroups of G_δ contained in Q_δ will be analyzed extensively. We begin with Z_δ which reappears over and over again in our arguments and is defined by

$$Z_\delta = \langle \Omega_1(Z(G_{\delta\lambda})) \mid \lambda \in \Delta(\delta) \rangle.$$

Letting $k \in \mathbb{N}$ we further define

$$G_\gamma^{[k]} = \langle Z_\lambda \mid \lambda \in \Delta^{[k]}(\delta) \rangle.$$

We shall concentrate on these subgroups mostly for $k \leq 4$, and we use the following abbreviations

$$V_\delta = G_\delta^{[1]},$$

$$U_\delta = G_\delta^{[2]}, \quad \text{and}$$

$$W_\delta = G_\delta^{[3]}.$$

The first phase of our investigations, which amounts to a very considerable step in pinning down the possible structure for G_{a_1} and G_{a_2}, consists of bouding the parameter b. This parameter, called the critical distance, is defined by

$$b = \min \{b_\mu \mid \mu \in V(\Gamma)\}$$

where

$$b_\mu = \min \{d(\mu, \lambda) \mid \lambda \in V(\Gamma),\, Z_\mu \not\leq Q_\lambda\}.$$

We note that $b \geq 1$. For $a, a' \in V(\Gamma)$, if $d(a, a') = b$ and $Z_a \not\leq Q_{a'}$, then we say that (a, a') is a critical pair and denote the set of critical pairs by $\mathcal{C}$. Suppose that $(\delta, \delta') \in \mathcal{C}$. Since Γ is a tree, there is a unique path in Γ between δ and δ' which we label in one of the following ways.

$$\begin{array}{ccccccccc}
\delta & & \delta+1 & & \delta+2 & & \delta+b-2 & \delta+b-1 & \delta' \\
\bullet & \!\!\text{———}\!\! & \bullet & \!\!\text{———}\!\! & \bullet & \cdots & \bullet \;\text{———} & \bullet & \text{———}\;\bullet \\
 & \delta'-b+1 & & & & & \delta'-2 & \delta'-1 &
\end{array}$$

Often we use $\delta - 1$ and $\delta' - 1$ to stand for, respectively, an arbitrary vertex of $\Delta(\delta) \setminus \{\delta + 1\}$ and $\Delta(\delta') \setminus \{\delta' - 1\}$. When using $(a, a') \in \mathcal{C}$ we shall always set $\beta = \alpha + 1$. Let $(\alpha, \alpha') \in \mathcal{C}$. If $[Z_\alpha, Z_{\alpha'}] \neq 1$, then we say that (α, α') is a non-commuting critical pair and if $[Z_\alpha, Z_{\alpha'}] = 1$, (α, α') is called a commuting critical pair.

This paper marks the beginning of our work on the commuting case for (S_3, S_6)-amalgams – the non-commuting case so far as bounding b, being covered in [LPR1]. Here we consider commuting critical pairs (α, α') with $\alpha \in O(S_6)$-those commuting critical pairs with $\alpha \in O(S_3)$ are the subject of parts V, VI and VII and the determinations of the structure of the (S_3, S_6)-amalgams once the critical distance in bounded can be found in [LPR2]. Our main conclusion here is contained in the following result.

THEOREM. *Suppose that for* $(\alpha, \alpha') \in \mathcal{C}$ *we have* $[Z_\alpha, Z_{\alpha'}] = 1$ *and* $\alpha \in O(S_6)$. *Then* $b \in \{1, 3\}$.

We conclude this introduction with some comments on the proof of this theorem as well as discussing some module facts.

Section 8 contains three preliminary results. One of these is the Core Argument given in Lemma 8.3. This is used frequently to lure subgroups into the G_α-cores of Z_α and Z^*_α (see Section 8 for the definition of Z^*_α). The structure of Z^*_α, dealt with in Lemma 8.3, is also important in many of our later deliberations.

The majority of Section 9 is taken up with the proof of Theorem 9.2 which proves that $[Z^*_\alpha : Z^*_\alpha \cap Z^*_{\alpha+2}] \neq 2$ so long as $b > 1$. Most of this proof is concerned with examining a particular Goldschmidt subamalgam (H_α, H_β) chosen so that it acts upon an FF-module V_0. This gives us access to results of Goldschmidt and Chermak which classify the possible amalgams and FF-modules. Our task then becomes the elimination of each of these possibilities. The most stubborn resistance is offered by the cases $\widehat{H} \cong S_6$ and $\widehat{H} \cong G_2(2)$ (where $H = \langle H_\alpha, H_\beta \rangle$ and $\widehat{H} = H/C_H(V_0)$). To deal with these cases we need to make use of another result of Chermak's [Ch2]. The last section of this paper investigates the case $[Z^*_\alpha : Z^*_\alpha \cap Z^*_{\alpha+2}] \neq 2$. Here we make greater use of the tree Γ. An important step in our analysis is Lemma 10.2 which says that our critical pairs are, in a certain sense, symmetric. Then in the next lemma we discover that, in fact, $Z^*_\alpha = Z_\alpha$. Lemma 10.4 observes certain facts about commutators and Lemma 10.5 describes some properties of $Sp_4(2)$ which are used in Lemma 10.6. Both Lemmas 10.7 and 10.8 focus upon U_α and, with these results to hand, the proof of the above theorem follows quickly.

If, for $\delta \in V(\Gamma)$, $M_\delta \leq N_\delta \leq Q_\delta$ with M_δ and N_δ both normal in G_δ, then $\eta(G_\delta, N_\delta/M_\delta)$ denotes the number of non-central G_δ-chief factors in

N_δ/M_δ. When looking at G_δ-invariant sections such as N_δ/M_δ we find that we need to know many details about irreducible $GF(2)$-modules for S_3 and S_6. The former group has just one non-trivial irreducible $GF(2)$-module which has dimension 2. For $H \cong S_6$ there are (up to isomorphism) four irreducible $GF(2)$-modules the most important for us being the two of dimension 4. Either of these modules will be referred to as a natural S_6-module (and sometimes denoted by 4). These two modules are related by the graph automorphism of $Sp_4(2) \cong S_6$. The 6-dimensional permutation module V for S_6 is indecomposable and has an irreducible composition factor of dimension 4. Calculations in V are easy and enable us to find out all we need to now about natural modules. Let U be a $GF(2)$ H-module. We call U an orthogonal module if U is indecomposable of dimension 5 and $U/C_U(H)$ is a natural module. Such a module is sometimes denoted by $\begin{pmatrix}4\\1\end{pmatrix}$ while $W \cong \begin{pmatrix}1\\4\end{pmatrix}$ indicates that W is an indecomposable module of dimension 5 with $[W,H]$ a natural module.

For an extensive (and up to date) account of amalgams the reader may consult [PR].

The authors thank Bernd Stellmacher for suggestions (too numerous to mention) which have substantially improved and shortened an earlier version of this paper.

8. Three Lemmata.

We begin by stating the following:

HYPOTHESIS 8.1. *If* $(\alpha,\alpha') \in \mathcal{C}$, *then* $[Z_\alpha, Z_{\alpha'}] = 1$, $\alpha \in O(S_6)$ *and* $b > 1$.

Before investigating the consequences of this hypothesis we state some well-known general observations on the commuting case.

LEMMA 8.2. *Let* $(\alpha,\alpha') \in \mathcal{C}$ *and assume* $[Z_\alpha, Z_{\alpha'}] = 1$.

(i) $b = b_\alpha$ *is odd and* $b_\beta = b+1$.

(ii) $Z_\beta = \Omega_1(Z(G_{\alpha\beta})) = \Omega_1(Z(G_\beta)) < Z_\alpha$, $Z(G_\alpha) = 1$ *and* $C_{G_\alpha}(Z_\alpha) = Q_\alpha$. *Also* $\eta(G_\alpha, Z_\alpha) \geq 1$ *and, if* $b > 1$, $\eta(G_{\alpha'}, V_{\alpha'}/Z_{\alpha'}) \geq 1$

If, additionally, $\alpha \in O(S_6)$, *then*

(iii) $G_{\alpha\beta} = Q_\alpha Q_\beta$; *and*

(iv) *if* $\Delta(\alpha') = \{\alpha'-1, \tau_1, \tau_2\}$, *then* $Z_{\alpha'-1}$, Z_{τ_1} *and* Z_{τ_2} *are pairwise distinct,* Z_α *is transitive on* $\{\tau_1,\tau_2\}$ *and* Z_{τ_1} *and* Z_{τ_2} *are conjugate by an element of* Z_α.

PROOF. Since $G_{\alpha'}/Q_{\alpha'} \cong S_3$ or S_6 and $Z_\alpha \not\leq Q_{\alpha'}$, $G_{\alpha'} = G_{\alpha'\,\alpha'-1}C_{G_{\alpha'}}(Z_{\alpha'})$ whence $Z_{\alpha'} = \Omega_1(Z(G_{\alpha'\alpha'-1})) \leq Z(G_{\alpha'})$. Then $Z_{\alpha'} = \Omega_1(Z(G_{\alpha'\alpha'-1})) = = \Omega_1(Z(G_{\alpha'}))$. If $b \equiv 0(2)$, then $Z_\alpha = \Omega_1(Z(G_{\alpha\beta})) \leq Z_\beta \leq Q_{\alpha'}$, a contradiction. So $b \equiv 1(2)$ and (i) holds. Suppose $b > 1$ and $\eta(G_{\alpha'}, V_{\alpha'}/Z_{\alpha'}) = 0$. Then, as $Z_{\alpha'} \leq Z_{\alpha'-1} \leq V_{\alpha'}$, we obtain $O^2(G_{\alpha'}) \leq N_{G_{\alpha'}}(Z_{\alpha'-1})$, against Lemma 1.1 (ii). Thus $\eta(G_{\alpha'}, V_{\alpha\prime}/Z_{\alpha'}) \geq 1$; the remainder of (ii) is a consequence of Lemma 1.1 (ii).

Clearly $Q_{\alpha'-1} \not\leq Q_{_{\alpha'}}$ and so as $\alpha' \in O(S_3)$ by (i) and $\alpha \in O(S_6)$ we have (iii). We note that $G_{\alpha'}$ is generated by two distinct edge stabilizers and so (iv) follows easily. □

For the rest of this paper we assume Hypothesis 8.1 holds (with the exception of Lemma 10.2 where we do not require $b > 1$). Let $(\alpha, \alpha') \in \mathcal{C}$. We choose Z^*_α to be a minimal normal subgroup of G_α contained in Z_α. By Lemma 8.2 (ii), Z^*_α is an irreducible G_α/Q_α-module. Now for $\tau \in O(S_6)$, we define subgroups $Z^*_\tau := (Z^*_\alpha)^g$, where $g \in G$ is such that $\alpha \cdot g = \tau$. This is easily seen to be well-defined. For $\lambda \in O(S_3)$ and $\tau \in \Delta(\lambda)$ we also define $V^*_\lambda = \langle Z^{*\,G_\lambda}_\tau \rangle$, $Y^*_\tau = \mathrm{core}_{G_\tau}(Z^*_\lambda)$ and $Y_\tau = \mathrm{core}_{G_\tau}(Z_\lambda)$.

LEMMA 8.3. *(The Core Argument) Let* $\tau \in O(S_3)$ *and* $\Delta(\tau) = \{\lambda_1, \lambda_2, \lambda_3\}$. *Suppose that* $H \leq G_{\lambda_1\tau}$ *and* $H \not\leq Q_\tau$. *Then*

(i) $Y^*_\tau = Z^*_{\lambda_i} \cap Z^*_{\lambda_j}$ *and* $Y_\tau = Z_{\lambda_i} \cap Z_{\lambda_j}$, *for* $i \neq j \in \{1, 2, 3\}$;

(ii) *H is transitive on* $\{\lambda_2, \lambda_3\}$; *and*

(iii) *if H normalizes a subgroup* M^* *of* $Z^*_{\lambda_i}$ *(respectively M of* Z_{λ_i}*) for some* $i \in \{2, 3\}$, *then* $M^* \leq Y^*_\tau$ *(respectively* $M \leq Y_\tau$*).*

PROOF. Note that $G_{\tau\lambda_k}$ normalizes $Z^*_{\lambda_i} \cap Z^*_{\lambda_j}$ and $Z_{\lambda_i} \cap Z_{\lambda_j}$ where $\{i, j, k\} = \{1, 2, 3\}$. So, since $[Z^*_{\lambda_i}, Q_{\lambda_i}] = [Z_{\lambda_i}, Q_{\lambda_i}] = 1$, $Z^*_{\lambda_i} \cap Z^*_{\lambda_j}$ and $Z_{\lambda_i} \cap Z_{\lambda_j}$ are both normalized by $\langle Q_{\lambda_i}, G_{\tau\lambda_k} \rangle = G_\tau$, and (i) holds. Part (ii) is clear while (iii) follows from (i) and (ii). □

LEMMA 8.4. *Let* $(\alpha, \alpha') \in \mathcal{C}$.

(i) *If* $[Z^*_\alpha : Z^*_\alpha \cap Z^*_{\alpha+2}] = 2$, *then* $\eta(G_\beta, V^*_\beta) = 1$.

(ii) $Q_\alpha \notin \mathrm{Syl}_2(\langle Q^{G_\beta}_\alpha \rangle)$.

(iii) Z^*_α *is natural* G_α/Q_α*-module.*

PROOF. (i) Assume that $[Z^*_\alpha : Z^*_\alpha \cap Z^*_{\alpha+2}] = 2$. Then $[Z^*_\alpha : Y^*_\beta] = 2$. By Lemma 1.1 (ii) $Z^*_\alpha < \langle Z^{*\,G_\beta}_\alpha \rangle = V^*_\beta$ and so, as $|V^*_\beta/Y^*_\beta| \leq 2^3$, $\eta(G_\beta, V^*_\beta) = 1$.

(ii) Suppose that $Q_\alpha \in \mathrm{Syl}_2(\langle Q^{G_\beta}_\alpha \rangle)$. From Lemma 8.2 (iii) $X_\beta := \langle Q^{G_\beta}_\alpha \rangle$ covers G_β/Q_β. Set $Q = Q_\alpha \cap Q_\beta$. Then $Q = O_2(X_\beta)$. Now Lemma 1.1 implies that no non-trivial characteristic subgroup of Q_a is normal in X_β. Appealing to 3.2 yields that

(8.4.1) $\eta(X_\beta, Q) = 1$ *and*

$$\langle \Omega_1(Z(Q))^x \mid x \in Aut(Q_a) \text{ and } x \text{ has odd order} \rangle$$

is a normal subgroup of X_β contained in Q.

Noting that $Q = \mathrm{core}_{G_\beta}(Q_a)$, we see that Q centralizes $\langle Z_a^{G_\beta} \rangle = V_\beta$. Hence, as $b \geq 3$, $V_\beta \leq \Omega_1(Z(Q))$. So (8.4.1) implies that $U_a = \langle V_\beta^{*O^2(G_a)} \rangle \leq Q$. Since $\eta(X_\beta, Q) = 1 = \eta(X_\beta, V_\beta)$ and $V_\beta \leq U_a \leq Q_a$, we have U_a is normal in X_β which is impossible by Lemma 1.1 (ii). Thus $Q_a \notin \mathrm{Syl}_2(\langle Q_a^{G_\beta} \rangle)$.

(iii) First we consider the case when $\eta(G_\beta, V_\beta^*) \geq 2$. Then $V_{a'}^* \cap Q_\beta \not\leq Q_a$. Since $(a, a') \in \mathcal{C}$ and $a' \in O(S_3)$, Z_a acts transitively on $\Delta(a') \setminus \{a' - 1\} = \{\tau, \rho\}$ and hence $Z_a^* \cap Q_\rho \leq C_{G_{a'}}(V_{a'}^*)$. So we have

(8.4.2) $$[Z_a^* : C_{Z_a^*}(V_{a'}^* \cap Q_\beta)] \leq [Z_a^* : Z_a^* \cap Q_\beta] \leq 2^4.$$

Now Proposition 2.9 (i) and (8.4.2) show that Z_a^* is not isomorphic to the 16-dimensional irreducible $GF(2)S_6$-module, whence Z_a^* is a natural module by Lemma 2.2 (i). So we may suppose that $\eta(G_\beta, V_\beta^*) = 1$. Thus $[V_\beta^*, Q_\beta] = [Z_a, Q_\beta]$. If Z_a^* is the 16-dimensional Steinberg module for $\bar{G}_a = G_a/Q_a$, then, by Proposition 2.9, $[Z_a^*, Q_\beta] \cong E(2^{15})$ and $C_{Z_a^*}(j) \cong E(2^8)$ for each involution j of $\bar{G}_a$. Therefore, $C_{G_{a\beta}}([V_\beta^*, Q_\beta]) = Q_a$. Set $H_\beta = C_{G_\beta}([V_\beta^*, Q_\beta])$. Then $H_\beta \trianglelefteq G_\beta$ and so $Q_a \in \mathrm{Syl}_2(H_\beta)$. But, as $H_\beta = \langle Q_a^{G_\beta} \rangle$, this contradicts part (ii), so completing the proof of (iii). $\square$

9. The case $[Z_a^* : Z_a^* \cap Z_{a+2}^*] = 2$.

Suppose that $(a, a') \in \mathcal{C}$. Our main result in this section is Theorem 9.2 in which we show that the case $[Z_a^* : Z_a^* \cap Z_{a+2}^*] = 2$ cannot occur. If $[Z_a^* : Z_a^* \cap Z_{a+2}^*] = 2$, then from Lemma 8.4 $\eta(G_\beta, V_\beta^*) = 1$ and $Z_a^* \cong E(2^4)$ is a natural G_a/Q_a-module. Moreover we have that $Y_\beta^* = [V_\beta^*, Q_\beta] = [Z_a^*, Q_\beta] \cong E(2^3)$. We shall use these facts without further references in this section.

Set $C_\beta = C_{G_\beta}(V_\beta^*)$, $K_\beta = \langle [V_\beta^*, Q_a]^{G_\beta} \rangle$ and $H_\beta = C_{G_\beta}(Y_\beta^*)$. Observe that $H_\beta = \langle Q_a^{G_\beta} \rangle$. Our first lemma looks at the structure of K_β and V_β^*.

LEMMA 9.1. *Assume that* $[Z_a^* : Z_a^* \cap Z_{a+2}^*] = 2$. *Then* $|K_\beta| = 2^3$, $K_\beta \cap Z_a^* = C_{Z_a^*}(G_{a\beta})$ *and* $K_\beta = [V_\beta^*, O^2(H_\beta)]$. *In particular,* $|V_\beta^*| = 2^6$.

PROOF. Since $[Z_\alpha^* : Y_\beta^*] = 2 = [Z_{\alpha+2}^* : Y_\beta^*]$, we have $[Z_{\alpha+2}^*, Q_\alpha \cap Q_\beta] \leq \leq Z_\beta \cap Z_\alpha^*$. Therefore, Q_α acts as a group of order 2 on $V_\beta^*/(Z_\beta \cap Z_\alpha^*)$ which centralizes the subgroup $[V_\beta^*, Q_\alpha] Z_\alpha^*/(Z_\beta \cap Z_\alpha^*)$. Since this latter group has index 2 in $V_\beta^*/(Z_\beta \cap Z_\alpha^*)$ and $[V_\beta^*, Q_\alpha] \not\leq Y_\beta^*$, we have $|\,[V_\beta^*, Q_\alpha](Z_\beta \cap Z_\alpha^*)| = 2^2$. If K_β has order 2^2, then $K_\beta Z_\alpha = V_\beta^*$ which then gives $[V_\beta^*, Q_\alpha] = 1$, contrary to $\eta(G_\beta, V_\beta^*) = 1$. Thus K_β has order 2^3 with $|K_\beta Y_\beta^*/Y_\beta^*| = 2^2$. Suppose $V_\beta^* = K_\beta Y_\beta^*$. Then $[V_\beta^*, Q_\alpha] \leq Z_\alpha^*$ and, as $[V_\beta^*, Q_\alpha]$ in normal in $G_{\alpha\beta}$, the uniseriality of Z_α^* as a $G_{\alpha\beta}$-module, implies that $[V_\beta^*, Q_\alpha] \leq Y_\beta^*$ which is of course impossible. Thus $|V_\beta^*/Y_\beta^*| = 2^3$ and $K_\beta \cap Z_\alpha^* = K_\beta \cap Y_\beta^* = Z_\alpha^* \cap Z_\beta = = C_{Z_\alpha^*}(G_{\alpha\beta})$.

THEOREM 9.2. *If $(\alpha, \alpha') \in \mathcal{C}$, then $[Z_\alpha^* : Z_\alpha^* \cap Z_{\alpha+2}^*] \neq 2$.*

PROOF. Suppose that $[Z_\alpha^* : Z_\alpha^* \cap Z_{\alpha+2}^*] = 2$ for $(\alpha, \alpha') \in \mathcal{C}$ and put $Q = Q_\alpha \cap Q_\beta$. Then we have that H_β has a Sylow 2-subgroup T where $\mathbb{Z}_2 \cong T/Q_\alpha \leq Z(G_{\alpha\beta}/Q_\alpha)$ with T/Q_α acting as a transvection on Z_α^*. Let $t \in O_2(H_\beta) \setminus Q$. Then $O_2(H_\beta) = Q\langle t\rangle$ and $T = \langle t\rangle Q_\alpha$.

We identify $G_{\alpha\beta}/Q_\alpha$ with $\langle (5\,6), (1\,3)(2\,4), (1\,2)\rangle$ and assume that Z_α^* is the natural module which admits $(5\,6)$ as a transvection. Then $tQ_\alpha = (5\,6)$. Let $I = \{1, 2, 3, 4\}$ and let $d_i \in G_\alpha$, $i \in I$, be such that

$$d_1Q_\alpha = (1\,5\,6),\ d_2Q_\alpha = (2\,5\,6),\ d_3Q_\alpha = (3\,5\,6),\ d_4Q_\alpha = (4\,5\,6).$$

Then tQ_α inverts d_iQ_α for each $i \in I$. Set $\mathcal{I} = \{\langle d_iQ_\alpha\rangle | i \in I\}$. For $i \in I$ we define the following subgroups:

$$\begin{aligned} H_{i,\alpha} &= \langle T, d_i\rangle;\\ H_i &= \langle H_\beta, H_{i,\alpha}\rangle;\\ N_i &= \mathrm{core}_{H_i}(T);\\ V_i &= \Omega_1(Z(N_i)) \text{ and}\\ U_i &= [V_i, H_i]\,. \end{aligned}$$

Note that for each $i \in I$, $O_2(H_{i,\alpha}) = Q_\alpha$ and that $H_{i,\alpha}/Q_\alpha \cong S_3$. For $i \in I$, we additionally set

$$\widetilde{H}_i = H_i/N_i,\ \widetilde{H}_{i,\alpha} = H_{i,\alpha}/N_i \quad \text{and} \quad \widetilde{H}_{i,\beta} = H_\beta/N_i\,.$$

Obviously we have

(9.2.1) *for $i \in I$, $\langle H_{i,\alpha}, G_{\alpha\beta}\rangle = G_\alpha$.*

Since $H_\beta/O_2(H_\beta) \cong S_3$ and $H_{i,a}/Q_a \cong S_3$, $\widetilde{H_i} = \widetilde{H_{i,a}} *_{\tilde{T}} \widetilde{H_{i,\beta}}$ is a Goldschmidt amalgam. Because $G_{a\beta}/Q_a$ permutes the set $\{\langle d_i Q_a\rangle \mid i \in I\}$ transitively and normalizes H_β, the type of the Goldschmidt amalgam is independent of $i \in I$.

Note that $N_i \leq O_2(H_{i,a}) \cap O_2(H_\beta) = Q_a \cap O_2(H_\beta) = Q$ and so, by Lemma 8.4 (ii),

(9.2.2) $\quad O_2(O^2(H_\beta)) \not\leq Q_a$.

Let $i \in I$. Recalling that t acts as a central transvection on Z_a^*, we have $Z_i := C_{Z_a^*}(\langle d_i, t\rangle) = C_{Z_a^*}(d_i) \cong E(2^2)$ with $Z_i \leq C_{Z_a^*}(t) = Y_\beta^*$. Hence, $Z_i \leq Z(H_i)$ and so $Z_i \leq N_i$. Therefore, $|\widetilde{Z_a^*}| \leq 2^2$. We next show that

(9.2.3) $\quad Z_a^* \leq N_i$ *for each* $i \in I$.

We suppose that (9.2.3) is false, and seek a contradiction. If $|Z_a^* N_i/N_i| = 2$, then $Z_a^* = \langle (Z_a^* \cap N_i)^{H_{i,a}}\rangle \leq N_i$. So we must have $|Z_a^* N_i/N_i| = 2^2$ with $\eta(\widetilde{H}_{i,a}, \widetilde{Z_a^*}) = 1$. Also, from (9.2.2), $\eta(\widetilde{H_{i,\beta}}, O_2(\widetilde{H_{i,\beta}})) \geq 1$. Let $\tilde{b}$ be the critical distance of the amalgam $(\widetilde{H_{i,a}}, \widetilde{H_{i,\beta}})$. Then $b \geq 3$ forces $\tilde{b} \geq 3$ which then yields, using Theorem 3.5, $\tilde{b} \geq 3$ and $\widetilde{H}_i$ is of type G_5 or G_5^1. Both of these amalgams have the property that $|Z(O^2(\widetilde{H_{i,\beta}}))| = 2$. Now $\tilde{b} = 3$ means that there exists $h \in \widetilde{H_i}$ such that $\widetilde{Z_a^*} \leq \widetilde{H}_{i,\beta}^h$ but $\widetilde{Z_a^*} \not\leq O_2(\widetilde{H}_{i,\beta}^h)$.

Since $[N_i, Z_a^*] = 1$, this leads to $[N_i, O^2(H_\beta)] = 1$. Hence using (9.2.2),

$$[N_i, \langle O^2(H_\beta),\ O^2(H_{i,a})\rangle] = 1\,.$$

Combining (9.2.1) and Lemma 1.1 (ii) gives

(9.2.3.1) $\quad N_i$ *contains no non-trivial* $G_{a\beta}$*-invariant subgroups.*

Let $g \in G_{a\beta}$. Since $G_{a\beta}$ normalizes $O^2(H_\beta)$, $[N_i^g, O^2(H_\beta)] = 1$ and so, as $|Z(O^2(\widetilde{H}_{i,\beta}))| = 2$, it follows that $N_i N_i^g \leq N_i Z_a^*$. Therefore $N_i Z_a^* \trianglelefteq G_{a\beta}$ and consequently $N_i Z_a^* \trianglelefteq G_a$ by (9.2.1). Because Q_a normalizes N_i, $[N_i Z_a^*, Q_a] = [N_i, Q_a]$ is a normal subgroup of G_a contained in N_i. So $[N_i Z_a^*, Q_a] = 1$ by (9.2.3.1). Clearly, $\Phi(N_i Z_a^*) \leq N_i$ and thus, using (9.2.3.1) again, we obtain $N_i Z_a^* \leq \Omega_1(Z(Q_a))$. Since $Q_a/N_i Z_a^*$ is a G_a-invariant section of Q_a with $\eta(H_{i,a}, Q_a/N_i Z_a^*) \neq 0$, $\eta(G_a, Q_a/N_i Z_a^*) \neq 0$. Now, by Theorem 3.5, $|\widetilde{T}| \leq 2^7$ which then shows that $Q_a/N_i Z_a^* \cong E(2^4)$ is a natural G_a/Q_a-module. As $[V_\beta^*, Q_a] \neq 1$, $V_\beta^* \not\leq N_i Z_a^*$. Hence Q_a is generated by involutions and thus, appealing to Lemma 3.11, Q_a is elementary abelian. But then Q_a/N_i is elementary abelian which is not the case as Q_a/N_i contains subgroups isomorphic to $\mathbb{Z}_4 \times \mathbb{Z}_4$. With this contradiction we have verified (9.2.3).

From (9.2.3), $Z_a^* \le V_i$ and therefore $V_\beta^* = \langle Z_a^{*H_\beta} \rangle \le V_i$ for all $i \in I$. Since

$$N_i \le C_{H_\beta}(V_i) \le C_\beta \le Q \le T \quad \text{and} \quad N_i \le C_{H_{i,a}}(V_i) \le C_{H_{i,a}}(Z_a^*) \le Q_a \le T\,,$$

we deduce that $C_{H_{i,a}}(V_i) = C_{H_\beta}(V_i)$. Hence $N_i = C_{H_{i,a}}(V_i) = C_{H_\beta}(V_i)$. Because $\langle H_{i,a}, G_{a\beta} \rangle = G_a$ by (9.2.1), $H_{i,a}$ does not normalize V_β^* and so $C_{H_i}(V_i)H_{i,a} \ne C_{H_i}(V_i)H_\beta$. Consequently, putting $\widehat{H_i} = H_i/C_{H_i}(V_i)$, $\widehat{H}_i = \widehat{H}_{i,a} *_{\widehat{T}} \widehat{H}_\beta$ is a Goldschmidt amalgam with $\widehat{H}_{i,a} \cong \widetilde{H}_{i,a}$ and $\widehat{H}_\beta \cong \widetilde{H}_{i,\beta}$.

The first part of the next claim has been discussed a moment ago.

(9.2.4) *For $i \in I$,*

(i) $V_\beta^* \le V_i$ *and* $N_i \le C_\beta$*; and*

(ii) $J(C_\beta) \not\le N_i$ *($J(C_\beta)$ being the elementary abelian version of the Thompson subgroup of C_β).*

For (ii) note that $J(C_\beta) \le N_i$ implies that $J(C_\beta) = J(N_i)$ is normalized by $\langle H_i, G_\beta \rangle = G$ and, as $J(C_\beta) \ne 1$, this contradicts Lemma 1.1.

It follows from (9.2.4) (ii) that $\widehat{J(C_\beta)}$ contains an offender on V_i. Further, as $\eta(\widehat{H}_\beta, O_2(\widehat{H})) \ge 1$ and $J(C_\beta) \trianglelefteq H_\beta$, Theorem 3.6 ([Table II and Corollary 3, Ch1]) yields that $\widehat{H_i} \cong S_6$ or $G_2(2)$ and all the following additional information.

(9.2.5) *For $i \in I$*

(i) $(\widetilde{H}_{1,a}, \widetilde{H}_{i,\beta})$ *is of type G_3^1 or G_4^1 and is independent of $i \in I$;*

(iI) $\widehat{H_i} \cong S_6$ *or* $G_2(2)$*;*

(iii) *if $\widehat{H_i} \cong S_6$, then U_i is isomorphic to either the natural or orthogonal S_6 module and C_β/N_i acts as a transvection on U_i; and*

(iv) *if $\widehat{H_i} \cong G_2(2)$, then $U_i/C_{U_i}(H_i)$ is isomorphic to the natural 6-dimensional $G_2(2)$-module and $|U_i| \le 2^7$. Furthermore, C_β/N_i has order 2^3 and is an offender on U_i.*

We will need the following result about amalgams of type G_3^1 or G_4^1.

(9.2.6) *Let $i \in I$ and suppose $\widetilde{W} \trianglelefteq \widetilde{H}_i$. If either $\widetilde{W} \cap \widetilde{H}_{i,a} \ne 1$ or $\widetilde{W} \cap \widetilde{H}_\beta \ne 1$, then $O^2(\widetilde{H}_i) \le \widetilde{W}$.*

If, say, $w \in \widetilde{W}$ where $\langle w \rangle \in \mathrm{Syl}_3(\widetilde{H}_\beta)$ then $[w, O_2(\widetilde{H}_\beta)] \le \widetilde{W}$. Since

$(\widetilde{H}_{i,a}, \widetilde{H}_\beta)$ is of type G_3^1 or G_4^1, $[w, O_2(\widetilde{H}_\beta)] \not\leq O_2(\widetilde{H}_{i,a})$ which forces $v \in \widetilde{W}$ where $\langle v\rangle \in \mathrm{Syl}_3(\widetilde{H}_{i,a})$. This yields $O^2(\widetilde{H}_i) \leq \widetilde{W}$. So we may suppose $\widetilde{W} \cap \cap \widetilde{H}_\beta \leq O_2(\widetilde{H}_\beta)$ and likewise that $\widetilde{W} \cap \widetilde{H}_{i,a} \leq O_2(\widetilde{H}_{i,a})$. But then $1 \neq \widetilde{W} \cap \cap \widetilde{H}_\beta = \widetilde{W} \cap \widetilde{H}_{i,a} \trianglelefteq \widetilde{H}_i$, a contradiction.

Note that $G_{a\beta}/Q_a$ permutes the set $\mathcal{I}$ transitively by conjugation. Assume that $g \in G_{a\beta}$ and $\langle d_i\, Q_a\rangle^g = \langle d_j Q_a\rangle$. Then, as $Q_a \leq T$, we have

$$H_{i,a}^g = H_{j,a}$$

and, because $G_{a\beta}$ normalizes H_β, we conclude that $H_i^g = H_j$, $N_i^g = N_j$, $V_i^g = V_j$ and $U_i^g = U_j$. Set, for $i, j \in I$ with $i \neq j$,

$$H_{ij} = \langle H_i, H_j\rangle$$

$$N_{ij} = \mathrm{core}_{H_{ij}}(T)$$

$$V_{ij} = \Omega_1(Z(N_{ij})) \quad \text{and}$$

$$\overline{H}_{ij} = H_{ij}/N_{ij}\,.$$

Note that $G_{a\beta}$ has two orbits on the pairs $\{i, j\} \subseteq I$. One orbit has length two and consists of $\{\{1,2\},\{3,4\}\}$ and the second orbit has length four and consists of all the other pairs.

Set $M = \bigcap_{i\in I} N_i$. Then M is normalized by $G_{a\beta}$. If $[N_1, d_i] \leq M$ were to hold then, $M \trianglelefteq \langle d_1, G_{a\beta}, H_\beta\rangle = G$ which yields $V_\beta^* \leq M = 1$. Thus $[N_1, d_1] \not\leq M$ and so there exist $j \in I \setminus \{I\}$ such that $[N_1, d_1] \not\leq N_j$. Since there is an element in $G_{a\beta}$ which exchanges 1 and j, we then have $[N_j, d_j] \not\leq N_1$. The action of $G_{a\beta}/Q_a$ on pairs in I shows that

(9.2.7) *there exists* $j \in \{2,3\}$ *such that* $[N_1, d_1] \not\leq N_j$.

Note that if $i, j \in I$, $i \neq j$, and $[N_i, d_i] \leq N_j$, then $N_{ij} = N_i \cap N_j$.

(9.2.8) *Suppose that* $j \in I$ *and* $[N_1, d_1] \not\leq N_j$. *Then*
(i) $C_{\overline{T}}(\overline{Q}_a) \leq \overline{Q}_a$; *and*
(ii) *for* $k \in \{1, j\}$, $C_{\overline{H}_\beta}(\overline{N}_k) \leq \overline{N}_k$, $C_{\overline{H}_{k,a}}(\overline{N}_k) \leq \overline{N}_k$, $\eta(\overline{H}_{k,a}, O_2(\overline{H}_{k,a})) \geq 2$ *and* $\eta(\overline{H}_\beta, O_2(\overline{H}_\beta)) \geq 2$.

If (i) is false, then d_1 would centralize $\overline{Q}_a$. Hence, since $N_{1j} \leq C_\beta \leq Q_a$, d_1 normalizes C_β whence $C_\beta \trianglelefteq \langle G_{a\beta}, d_1\rangle = G_a$, a contradiction. Thus (i) holds.

Suppose $C_{\overline{H}_\beta}(\overline{N}_k) \not\leq \overline{N}_k$ (where $k \in \{1, j\}$). Applying (9.2.6) with

$\widetilde{W} = \widetilde{C_{\overline{H}_k}(\overline{N}_k)}$ yields that d_k centralizes $\overline{N}_k$. So $[N_k, d_k] \leq N_{1j} \leq N_1 \cap N_j$ which is not the case. Therefore $C_{\overline{H}_\beta}(\overline{N}_k) \leq \overline{H}_k$ and likewise $C_{\overline{H}_{k,a}}(\overline{N}_k) \leq \overline{N}_k$ for $k = 1, j$. Now, appealing (9.2.5) (ii), the remainder of (ii) follows.

In the language of Chermak [Ch2], (9.2.7) and (9.2.8) say that, provided $[N_1, d_1] \not\leq N_j$, $\{T \subseteq H_{1,a}, H_{j,a}, H_\beta\}$ is a "Large Triangular Amalgam". Since $O_2(H_{1,a}) = Q_a = O_2(H_{j,a})$, we may apply [Theorem A, Ch2] to obtain the following result.

(9.2.9) *Assume that* $[N_1, d_1] \not\leq N_j$. *Then, for* $k \in \{1, j\}$, N_k/N_{1j} *is elementary abelian, contains exacitly one non-central* H_k*-chief factor and* $N_1 \cap N_j$ *is not normal in* H_k.

(9.2.10) *Suppose that* $[N_1, d_1] \not\leq N_j$. *Then* H_1 *and* H_j *have no non-central chief factors within* N_{1j}.

Suppose that $C_T(N_{1j}) \leq N_{1j}$. Then $V_1 V_j \leq V_{1j}$. Define

$$m = \min_{A \in \mathcal{A}(C_\beta)} \{|\overline{A}| \, A \not\leq N_1 \text{ or } A \not\leq N_2\}$$

and

$$\mathcal{K}(C_\beta) = \{B \in \mathcal{A}(C_\beta) \,|\, |\overline{B}| < m\} \,.$$

Then, by the definition of m, $\langle \mathcal{K}(C_\beta) \rangle \leq N_1 \cap N_j$ and consequently is invariant under the conjugation action of H_1 and H_j. Therefore, $\langle \mathcal{K}(C_\beta) \rangle \leq N_{1j}$. By (9.2.4) (ii), $J(C_\beta) \not\leq N_i$ for $i \in I$, so $m > 1$. Furthermore, as there is an element of $G_{\alpha\beta}$ which interchanges 1 and j we know that there is $A \in \mathcal{A}(C_\beta)$ with $A \not\leq N_1$ and $|\overline{A}| = m$. From among all such A select one with $|A \cap V_1|$ maximal. Then by the Thompson Replacement Theorem A is an offender on V_1. So, since $[C_\beta : N_1] = 2$ in the case $\widehat{H_1} \cong S_6$ and by [ON] when $\widehat{H_1} \cong G_2(2)$, we have

$$|AN_1/N_1| = |V_1/C_{V_1}(A)| = \begin{cases} 2 & \text{if } \widehat{H_1} \cong S_6 \\ 2^3 & \text{if } \widehat{H_1} \cong G_2(2) \end{cases}$$

In particular, this yields that $B = (A \cap N_1)V_1 \in \mathcal{A}(C_\beta)$. Clearly, as $V_1 \leq N_{1j}$, $\bar{B} \leq \bar{N}_1$ and $|\bar{B}| < |\bar{A}|$. Hence $B \in \mathcal{K}(C_\beta)$ and thus $B \leq N_{1j}$. In particular, $A \cap N_1 \leq N_{1j}$ and so

$$|V_{1j}/C_{V_{1j}}(A)| = |V_{1j}/V_{1j} \cap A| = |A/A \cap N_{1j}| = \begin{cases} 2 & \text{if } \widehat{H_1} \cong S_6 \\ 2^3 & \text{if } \widehat{H_1} \cong G_2(2) \,. \end{cases}$$

Therefore $V_{1j} = C_{V_{1j}}(A)V_1$. Since V_{1j}/V_1 admits H_1 and $A \not\leq N_1$, using (9.2.6) gives $V_1V_j \trianglelefteq H_1$ which contradicts (9.2.9). From this contradiction we deduce that $C_T(N_{1j}) \not\leq N_{1j}$. If $C_T(N_{1j}) \leq N_1 \cap N_j$, then $C_T(N_{1j}) = = C_{N_1}(N_{1j}) = C_{N_j}(N_{1j}) \trianglelefteq H$, whence $C_T(N_{1j}) \leq N_{1j}$. Thus $C_T(N_{1j}) \not\leq N_i$ where $i \in \{1, j\}$. Appealing to (9.2.6) we infer that $O^2(H_\beta)$ centralizes N_{1j} and then, by (9.2.6) again, we have proved (9.2.10).

One consequence of (9.2.5), (9.2.9) and (9.2.10) is that $U_1 \not\leq N_{1j}$ and that U_1N_{1j}/N_{1j} contains the unique non-central H_1-chief factor in N_1/N_{1j}.

(9.2.11) $Z_a^* = Z_a$ *and* $Z_\beta = C_{Z_a^*}(G_{a\beta})$.

Since every $H_{1,a}$-chief factor of N_1 is contained in U_1 and $\eta(H_{1,a}, C_{U_1}(Q_a)) = 1$, $\eta(H_{1,a}, \Omega_1(Z(Q_a))) = 1$. Hence, as $Z(G_a) = 1$, Lemma 2.2 implies that $\Omega_1(Z(Q_a))$ is either a natural or a dual orthogonal module for S_6. In particular, (9.2.11) holds.

(9.2.12) *Suppose that* $[N_1, d_1] \not\leq N_j$ *and* $\gamma = \beta.d_1 \in \Delta(a)$. *Then*

(i) $[Z_a, d_1]K_\beta \leq U_1$, $|[Z_a, d_1]K_\beta| = 2^4$ *and* $Z_a \cap [Z_a, d_1]K_\beta = [Z_a, d_1]$;

(ii) $[K_\gamma, C_\beta] \not\leq Z_a$; *and*

(iii) $U_1N_j = U_jN_1 = N_1N_j = C_\beta$.

Since $\eta(H_1, N_1) = 1$, $[Z_a, d_1] \leq [N_1, d_1] \leq U_1$. Now $[Z_a, d_1] = Z_\beta Z_\gamma$ and so $Z_\beta \leq U_1$. Now Lemma 9.1 implies that $K_\beta \leq [Z_a, d_1]^{H_\beta} \leq U_1$ and that $|[Z_a, d_1]K_\beta| = 2^4$. So (i) holds.

Because $\eta(G_\gamma, K_\gamma) = 1$ and $C_\beta \not\leq Q_\gamma$ from the structure of the amalgam $(H_{1,a}/N_1, H_\beta/N_1)$, part (ii) holds. Suppose that $U_1 \leq N_j$. Then $[N_1, d_1] \leq \leq U_1 \leq N_j$, a contradiction. Hence (iii) holds by (9.2.5) (iii) and (iv).

(9.2.13) $\widehat{H_1} \not\cong S_6$.

Suppose that $\widehat{H_1} \cong S_6$. Then $[U_1, C_\beta]$ has order 2. Assume that $[N_1, d_1] \not\leq N_3$. Then, by (9.2.12) (iii) and by symmetry,

$$[U_3, C_\beta] = [U_1, U_3] = [U_1, C_\beta].$$

Since also $[U_1, d_1] \not\leq N_4$, we have

$$[U_1, C_\beta] = [U_1, U_4] = [U_4, C_\beta].$$

Therefore, $[U_1, C_\beta]$ is normalized by $\langle(1\ 4)(2\ 3), (3\ 4), (5\ 6)\rangle = G_{a\beta}/Q_a$. Hence, as $|[U_1, C_\beta]| = 2$, $[U_1, C_\beta] = Z_\beta$ by (9.2.11) and this contradicts (9.2.12) (ii).

Next suppose $[N_1, d_1] \not\leq N_2$ and that $[N_1, d_1] \leq N_3$. Applying (1 3)(2 4) to $[N_1, d_1] \not\leq N_2$ gives $[N_3, d_3] \not\leq N_4$. If $U_3 \not\leq N_1$, then, as $|C_\beta/N_1| = 2$, $[U_1, U_2] = [U_1, C_\beta] = [U_1, U_3]$ is normalized by $\langle (1\ 2), (1\ 3)(2\ 4), (5\ 6) \rangle = G_{\alpha\beta}/Q_\alpha$ and we have the same contradiction as above. Therefore, $U_3 \leq N_1$ and so $U_3 U_1$ is normalized by H_1 and, recalling that $[N_3, d_3] \not\leq N_4$,

$$[U_3, N_1] \leq [U_3, U_4] .$$

If $[U_3, N_1] = 1$, then $N_3 = N_1$ and so from the action of $G_{\alpha\beta}$ on I we get $N_1 = N_2 = N_3 = N_4 = 1$, which is impossible. Therefore, $[U_3, N_1] = [U_3, U_4]$ and conjugating by (1 2) we have $[U_3, U_4]$ is normalized by $\langle (3\ 4), d_1, d_2 \rangle \cong S_4 \times 2$. In particular, $[U_1, U_4] \leq Z_\alpha$ and we have a contradiction to (9.2.12) (ii). We conclude that $\widehat{H_1} \ncong S_6$.

By (9.2.5) (ii) and (9.2.13), to complete the proof of Theorem 9.2 it remains to eliminate the possibility $\widehat{H_1} \cong G_2(2)$. So assume that $\widehat{H_1} \cong G_2(2)$ and that $[N_1, d_1] \not\leq N_j$. Then from [ON], $[U_1 : C_{U_1}(C_\beta)] = 2^3$ and so $|C_{U_1}(C_\beta)| \leq 2^4$ by (9.2.5) (iv). Now (9.2.12) (ii) implies that $C_{U_1}(C_\beta) = K_\beta[Z_\alpha, d_1]$. Set $\gamma = \beta.d_1$. Then $1 \neq [K_\gamma, C_\beta] = [K_\gamma, U_j]$. Since U_j does not admit transvections from $\widehat{H_j}$, we have that $|[K_\gamma, C_\beta]| = 2^2$ by Lemma 9.1. It follows that $Z_\gamma \leq [U_1, U_j]$ and hence

$$C_{U_1}(U_j) = [U_1, U_j] = C_{U_j}(U_1) .$$

But then $[Z_\alpha, d_j][Z_\alpha, d_1] \leq C_{U_1}(C_\beta)$ and this contradicts $|Z_\alpha \cap C_{U_1}(C_\beta)| = 2^2$. $\square$

10. Determination of b.

Again, let $(\alpha, \alpha') \in \mathcal{C}$. From Theorem 9.2 we have that $[Z_\alpha^* : Z_\alpha^* \cap Z_{\alpha+2}^*] > 2$. In this section we show that this situation leads to $b \leq 3$, so establishing our main theorem. So, by Lemma 8.3 (i), we have $[Z_\alpha^* : Y_\beta^*] > 2$. First we observe

Lemma 10.1. *Suppose $\tau \in O(S_3)$ and $\lambda \in \Delta(\tau)$. Then no element in $G_\tau \setminus G_\lambda$ centralizes a hyperplane of Z_λ^*.*

Proof. This follows from $[Z_\alpha^* : Y_\beta^*] > 2$ and Lemma 8.3 (iii). $\square$

By Lemma 8.4 (iii), Z_α^* is a natural G_α/Q_α-module-this fact will be used without reference. The next lemma is also needed when we later investigate the $b = 1$ and $b = 3$ cases.

LEMMA 10.2. *If* $b \geq 1$, *then*

(i) *for each* $\alpha' + 1 \in \Delta(\alpha') \setminus \{\alpha' - 1\}$, $Z^*_{\alpha'+1} \not\leq Q_\beta$;

(ii) $V^*_{\alpha'} \not\leq Q_\beta$; *and*

(iii) $Z^*_\alpha \not\leq Q_{\alpha'}$.

PROOF. First we prove part (i), so we let $\alpha' + 1 \in \Delta(\alpha') \setminus \{\alpha' - 1\}$. Assume that $Z^*_{\alpha'+1} \leq Q_\beta$, and argue for a contradiction. Since $Z_\alpha \not\leq Q_{\alpha'}$, Lemma 10.1 gives

$$[Z^*_{\alpha'+1} : Z^*_{\alpha'+1} \cap Q_\alpha] \geq 4\,.$$

So $|Z^*_{\alpha'+1}Q_\alpha/Q_\alpha| \geq 4$ and hence $[Z^*_\alpha/(Z^*_\alpha \cap Q_{\alpha'+1})] \geq [Z^*_\alpha : C_{Z^*_\alpha}(Z^*_{\alpha'+1})] \geq 4$ because a fours subgroup of S_6 does not centralize any hyperplane of the natural module (see Proposition 2.5).

Set $R = [Z^*_\alpha \cap Q_{\alpha'}, Z^*_{\alpha'+1}]$. Clearly, $R \leq Z^*_\alpha \cap Z^*_{\alpha'+1}$. Since $[Z^*_\alpha : C_{Z^*_\alpha}(Z^*_{\alpha'+1})] \geq$ ≥ 4, $R \neq 1$. Also $R \leq Z^*_\alpha \cap Z^*_{\alpha'+1} \leq C_{Z^*_{\alpha'+1}}(Z_\alpha)$ gives $R \leq Z^*_{\alpha'-1} \cap Z^*_{\alpha'+1} = Y^*_{\alpha'}$, by Lemma 8.3. Hence $|R| = 2$ or 4. First we examine the case $|R| = 2$. If $Z^*_\alpha \leq Q_{\alpha'}$, then we get $[Z^*_\alpha : Z^*_\alpha \cap Q_{\alpha'+1}] \leq 2$ whereas $[Z^*_\alpha : Z^*_\alpha \cap Q_{\alpha'+1}] \geq 4$. So $Z^*_\alpha \not\leq Q_{\alpha'}$. Now choose $t \in Z^*_\alpha \setminus Q_{\alpha'}$. Looking at $Z^*_{\alpha'+1}$ acting on Z^*_α we have that $Z^*_{\alpha'+1}$ leaves R invariant and, by Proposition 2.5 (ii), $|[Z^*_\alpha, Z^*_{\alpha'+1}]| \leq 4$. Hence $Z^*_{\alpha'+1}$ acts (quadratically) on the 3-space Z^*_α/R with $|[Z^*_\alpha/R, Z^*_{\alpha'+1}]| \leq 2$. As a consequence there exists $X \leq Z^*_{\alpha'+1}$ with $[Z^*_{\alpha'+1} : X] \leq 2$ and such that $[t, X] \leq R$. Therefore t normalizes XR, whence $XR \leq Y^*_{\alpha'}$ by Lemma 8.3 (iii) which contradicts $|Y^*_{\alpha'}| \leq 4$. Thus $|R| \neq 2$ and so we have $|R| = 4$. Because $|Z^*_{\alpha'+1}Q_\alpha/Q_\alpha| \geq 4$, $|[Z^*_\alpha, Z^*_{\alpha'+1}]| \geq 4$ by Proposition 2,5 (ii) which, as $|R| = 4$, forces $R = [Z^*_\alpha, Z^*_{\alpha'+1}]$. Therefore

$$[Z^*_\alpha, Z^*_{\alpha'+1}] = R \leq Z^*_{\alpha'-1} \cap Z^*_{\alpha'+1} = Y^*_{\alpha'}\,.$$

Thus Z^*_α normalizes $Z^*_{\alpha'+1}$ and so $Z^*_\alpha \leq Q_{\alpha'}$ by Lemma 8.3 (iii). Since

$$R \leq [Z_\alpha \cap Q_{\alpha'}, Z^*_{\alpha'+1}] \leq Z_\alpha \cap Z^*_{\alpha'+1} \leq C_{Z^*_{\alpha'+1}}(Z_\alpha)$$

and $Z_\alpha \not\leq Q_{\alpha'}$,Lemma 10.1 and $|R| = 4$ imply that

$$[Z_\alpha \cap Q_{\alpha'}, Z^*_{\alpha'+1}] = R \leq Z^*_\alpha\,.$$

So $Z^*_{\alpha'+1}$ centralizes a hyperplane of Z_α/Z^*_α and therefore, because $|Z_{\alpha'+1}Q_\alpha/Q_\alpha| \geq 4$, $\eta(G_\alpha, Z_\alpha/Z^*_\alpha) = 0$.

Because $Z^*_\alpha \leq Q_{\alpha'}$ and $(\alpha, \alpha') \in \mathcal{C}$, $Z^*_\alpha < Z_\alpha$. However, combining Lemmas 8.2 (ii) and 2.4, we then have $Z_\alpha \cong \begin{pmatrix} 1 \\ 4 \end{pmatrix}$. But by Proposition 2.6 (ii), $\langle C_{Z_\alpha}(G_{\alpha\beta})^{G_\alpha} \rangle \neq Z_\alpha$ contrary to the definition of Z_α. With this contradiction we have completed the proof of part (i).

Clearly part (i) gives part (ii). Also part (i) implies that for $\alpha' + 1 \in \Delta(\alpha') \setminus \{\alpha' - 1\}$, $(\alpha' + 1, \beta) \in \mathcal{C}$. Thus applying (i) to this critical pair we obtain that $Z^*_\alpha \not\leq Q_{\alpha'}$, so proving the lemma. $\square$

LEMMA 10.3.
(i) $Z_\alpha = Z^*_\alpha$ *is a natural* G_α/Q_α*-module.*
(ii) *For* $\lambda \in \Delta(\beta) \setminus \{\alpha\}$, $Y_\beta = Z_\alpha \cap Z_\lambda \cong E(2^2)$.

PROOF. We begin by establishing part (i). Pick $\alpha' + 1 \in \Delta(\alpha') \setminus \{\alpha' - 1\}$. Let $t \in Z^*_\alpha \setminus Q_{\alpha'}$. Then, using Lemma 8.3 (iii),

$$Y_{\alpha'} \leq Z_{\alpha'+1} \cap Z_{\alpha'-1} \leq Z_{\alpha'+1} \cap Q_\alpha \leq C_{Z_{\alpha'+1}}(t) \leq Y_{\alpha'} \,.$$

So, for every $t \in Z^*_\alpha \setminus Q_{\alpha'}, Z_{\alpha'+1} \cap Q_\alpha = C_{Z_{\alpha'+1}}(t)$. Set $B = Z_{\alpha'+1} \cap Q_\beta$. We claim that at most one element in BQ_α/Q_α acts as a transvection on Z^*_α. For suppose $t_1, t_2 \in B$ are such that t_1Q_α and t_2Q_α are distinct transvections on Z^*_α. Then without loss of generality $C_{Z^*_\alpha}(t_1) \neq Z^*_\alpha \cap Q_{\alpha'}$ and so we may find $t \in Z^*_\alpha \setminus Q_{\alpha'}$ with $t \in C_{Z^*_\alpha}(t_1)$. Hence $t_1 \in C_{Z_{\alpha'+1}}(t) = Z_{\alpha'+1} \cap Q_\alpha$. But then $[Z^*_\alpha, t_1] = 1$ whereas t_1 acts as a transvection on Z^*_α, so verifying the claim.

Since B acts quadratically on Z^*_α we infer that $|BQ_\alpha/Q_\alpha| \leq 4$. A symmetric argument yields that $|(Z_\alpha \cap Q_{\alpha'})Q_{\alpha'+1}/Q_{\alpha'+1}| \leq 4$. Then we conclude that $[Z_\alpha : Z_\alpha \cap Q_{\alpha'} \cap Q_{\alpha'+1}] \leq 2^3$ and so $[Z_\alpha : C_{Z_\alpha}(B)] \leq 2^3$.

We now show that $\eta(G_\alpha, Z_\alpha/Z^*_\alpha) = 0$. If $|BQ_\alpha/Q_\alpha| = 4$, then $[Z_\alpha : C_{Z_\alpha}(B)] \leq$ $\leq 2^3$ gives $\eta(G_\alpha, Z_\alpha/Z^*_\alpha) = 0$. So we may assume that $|BQ_\alpha/Q_\alpha| \leq 2$. By Lemma 10.2 (i) $G_{\beta\alpha+2} = Q_\beta Z^*_{\alpha'+1} = Q_\beta Z_{\alpha'+1}$ from which it follows that $Z_{\alpha'+1} = Z^*_{\alpha'+1}(Z_{\alpha'+1} \cap Q_\beta)$. Now Lemma 10.1 implies that $Z^*_{\alpha'+1} \cap Q_\beta \not\leq Q_\alpha$, whence $BQ_\alpha = (Z^*_{\alpha'+1} \cap Q_\beta)Q_\alpha$ from which we get $B = Z_{\alpha'+1} \cap Q_\beta = (Z^*_{\alpha'+1} \cap Q_\beta)$ $(Z_{\alpha'+1} \cap Q_\beta \cap Q_\alpha)$. So

$$Z_{\alpha'+1} = Z^*_{\alpha'+1}(Z_{\alpha'+1} \cap Q_\alpha \cap Q_\beta) \,.$$

Since $(\alpha' + 1, \beta) \in \mathcal{C}$, $Z_\alpha \cap Q_{\alpha'} \not\leq Q_{\alpha'+1}$ by Lemma 10.1 and hence, as $Z_\alpha \cap Q_{\alpha'}$ centralizes $Z_{\alpha'+1}/Z^*_{\alpha'+1}$, $\eta(G_{\alpha'+1}, Z_{\alpha'+1}/Z^*_{\alpha'+1}) = 0$.

From $\eta(G_\alpha, Z_\alpha/Z^*_\alpha) = 0$ we deduce that either $Z_\alpha = Z^*_\alpha$ or $Z_\alpha \cong \begin{pmatrix} 1 \\ 4 \end{pmatrix}$. The latter is impossible as, by Proposition 2.6 (ii), we have $\langle C_{Z_\alpha}(G_{\alpha\beta})^{G_\alpha} \rangle \neq Z_\alpha$. So we conclude that $Z_\alpha = Z^*_\alpha$, which establishes (i).

Moving on to part (ii), if $|(Z_{\alpha'+1} \cap Q_\beta)Q_\alpha/Q_\alpha| \geq 4$, then Lemma 10.2 (iii) and $Z_\alpha \not\leq Q_{\alpha'}$ forces $[Z_\alpha, Z_{\alpha'+1} \cap Q_\beta] \leq Z_\alpha \cap Z_\lambda$, which yields (ii). While, if $|(Z_{\alpha'+1} \cap Q_\beta)Q_\alpha/Q_\alpha| \leq 2$, then (ii) again follows from Lemma 10.2 (iii) since it gives $Z_{\alpha'+1} \cap Q_\beta \cap Q_\alpha \leq Y_{\alpha'}$ and $|Z_{\alpha'+1} \cap Q_\beta \cap Q_\alpha| \geq 4$. $\square$

LEMMA 10.4.
(i) $\eta(G_\beta, V_\beta) = 2$.
(ii) $|[V_\beta, V_{\alpha'}]| \geq 2^3$.
(iii) $|Y_\beta[V_\beta, Q_\alpha]| \leq 2^4$.

PROOF. Lemma 10.3 (ii) imediately gives $\eta(G_\beta, V_\beta) = 2$. Hence, as $V_{\alpha'} \not\leq Q_\beta$, $|[V_\beta/Y_\beta, V_{\alpha'}]| \geq 2^2$. From $1 \neq [V_{\alpha'} \cap Q_\beta, Z_\alpha] \leq Y_\beta$ we then see that $|[V_\beta, V_{\alpha'}]| \geq 2^3$.

Since $V_{\alpha'} \not\leq Q_\beta$, using the transitivity of G_β on $\Delta(\beta)$ we may find an involution $t \in Q_\alpha \setminus Q_\beta$. Thus $Q_\alpha = (Q_\alpha \cap Q_\beta)\langle t\rangle$ with t interchanging the vertices in $\Delta(\beta) \setminus \{\alpha\}$. Hence, $[V_\beta, Q_\alpha] = [V_\beta, Q_\alpha \cap Q_\beta][V_\beta, t]$ and $|[V_\beta, t]| = 2^2$. Because $[Y_\beta, Q_\alpha \cap Q_\beta] = 1$, $[V_\beta, Q_\alpha \cap Q_\beta] = [Z_{\alpha+2}, Q_\alpha \cap Q_\beta]\,[Z_\lambda, Q_\alpha \cap Q_\beta] \leq$ $\leq Y_\beta$, where $\Delta(\beta) = \{\alpha, \alpha+2, \lambda\}$. Therefore, as $|Y_\beta| = 2^2$, $|Y_\beta[V_\beta, Q_\alpha]| \leq$ $\leq 2^4$. □

The following property of $Sp_4(2)(\cong S_6)$ will be deployed in Lemma 10.6.

LEMMA 10.5. *Suppose $H \cong Sp_4(2)$ and that V is a natural $Sp_4(2)$-module for H. Then there exists a maximal subgroup L of H such that*
(i) *$L \cong S_5$;*
(ii) *L contains no transvections (on V);*
(iii) *V is a natural $SL_2(4)$-module for L'; and*
(iv) *there are 5 isotropic 2-subspaces $W_1, \ldots, W_5$ of V which are permuted by L and such that $V = \bigcup_{i=1}^{5} W_i$ and $W_i \cap W_j = 0$ for $i \neq j$.*

PROOF. Choose L to be a maximal subgroup of H with $L \cong S_5$ and so as for $\langle g\rangle \in \mathrm{Syl}_3(L)$, $C_V(g) = 0$. Then V is a natural $SL_2(4)$- module for L' and we also get (ii). Further $\{C_V(R) | R \in \mathrm{Syl}_2(L')\}$ is a partition of V consisting of 5 isotropic 2-subspaces of V (that they are isotropic follows from $\mathrm{Stab}_L(C_V(R)) \cong S_4$), and the lemma is proved. □

Since $|Y_\beta| = 4$ by Lemma 10.3 (ii), $Y_\beta = [Z_\alpha, G_{\alpha\beta}; 2]$ and so, by Proposition 2.8 (i), Y_β is an isotropic 2-subspace of Z_α. Now G_α acts transitively on the isotropic 2-subspaces of Z_α and therefore we may find a subgroup L_α of G_α such that $L_\alpha/Q_\alpha \cong S_5$ has the properties in Lemma 10.5 with Y_β a member of the partition given in Lemma 10.5 (iv).

Set $\widehat{U}_\alpha = \langle V_\beta^{L_\alpha}\rangle$. Now fix $\alpha - 1 = \beta^g$ where $g \in L_\alpha$ is such that $Y_{\alpha-1} \cap Y_\beta = 1$. Also choose, and fix, $\alpha' + 1 \in \Delta(\alpha') \setminus \{\alpha' - 1\}$. Since $|Y_\beta| = 2^2$, if $[X, Y_\beta] = 1$ and $X \leq Q_\beta$, then $[V_\beta, X] \leq Y_\beta$-we shall use this fact without further reference.

LEMMA 10.6. *If* $b \geq 5$, *then*
(i) $C_{Z_a}(Z_{a'+1}) = Y_\beta$ *and* $C_{Z_{a'+1}}(Z_a) = Y_{a'}$;
(ii) $Y_a \cap V_{a'} \neq 1$;
(iii) $V_{a-1} \leq Q_{a'-1}$; *and*
(iv) $Y_{a'} \neq Y_\beta$.

PROOF. Part (i) follows from $Z_{a'+1} \not\leq Q_\beta$ and Lemma 10.1. From part (i) and $|Y_{a'}| = 4$ we get that $Z_{a'+1} \cap Q_\beta \not\leq Q_a$. Thus

$$1 \neq [Z_a, Z_{a'+1} \cap Q_\beta] \leq Z_a \cap V_{a'} \leq C_{Z_a}(Z_{a'+1}) \cap V_{a'} = Y_\beta \cap V_{a'} ,$$

and so (ii) holds.

We next prove (iii). If there exists $a-2 \in \Delta(a-1)$ such that $(a-2, a'-2) \in \mathcal{C}$, then by part (ii) applied to $(a-2, a'-2)$ we have $Y_{a-1} \cap V_{a'-2} \neq 1$. Because $b \geq 5$ $[Y_{a-1} \cap V_{a'-2}, Z_{a'-1}] = 1$ and so $Y_{a-1} \cap V_{a'-2} \leq$ $\leq C_{Z_a}(Z_{a'+1}) = Y_\beta$ which is impossible as $Y_{a-1} \cap Y_\beta = 1$. Thus we conclude that $V_{a-1} \leq Q_{a'-2}$. We further deduce that $(a'-1, a-1) \notin \mathcal{C}$ as otherwise, using Lemma 10.2 (i), we get $(a-2, a'-2) \in \mathcal{C}$ for some $a-2 \in \Delta(a-1)$ whereas $V_{a-1} \leq Q_{a'-2}$. So $Z_{a'-1} \leq Q_{a-1}$. Clearly $Z_{a'-1}$ centralizes Y_{a-1} and hence $[V_{a-1}, Z_{a'-1}] \leq Y_{a-1}$. Also, as $V_{a-1} \leq Q_{a'-2} \leq G_{a'-1,}$ $[V_{a-1}, Z_{a'-1}] \leq Z_{a'-1}$. Therefore, using (i),

$$[V_{a-1}, Z_{a'-1}] \leq Y_{a-1} \cap Z_{a'-1} \leq Y_{a-1} \cap C_{Z_a}(Z_{a'+1}) = Y_{a-1} \cap Y_\beta = 1.$$

Consequently, $V_{a-1} \leq Q_{a'-1}$, as required.

Finally, assume that $Y_{a'} = Y_\beta$. Then $[Z_a \cap Q_{a'}, Z_{a'+1}] \leq Y_{a'} = Y_\beta$.Hence, $Z_{a'}$ normalizes $Z_a \cap Q_{a'}$ and so Lemmas 10.2 (i) and 8.3 (iii) force $Z_a \cap Q_{a'} \leq Y_\beta$, a contradiction. Therefore, $Y_{a'} \neq Y_\beta$. □

LEMMA 10.7. *Suppose that* $b \geq 5$ *and* $Y_{a'} \neq Y_{a'-2}$. *Then* $U_a \leq C_{G_{a'}}(Y_{a'})$. *In particular, if* $b \geq 5$, $[U_a \cap Q_{a'}, V_{a'}] \leq Y_{a'}$.

PROOF. From $Y_{a'} \neq Y_{a'-2}$ we see that $Y_{a'} Y_{a'-2}$ is a subgroup of $Z_{a'-1}$ of order at least 8. Suppose $[U_a, Y_{a'}] = 1$. Then $[U_a, Y_{a'} Y_{a'-2}] = 1$, whence $U_a \leq Q_{a'-2} \leq G_{a'-1}$ by Lemma 10.1. This then gives

$$U_a \leq C_{G_{a'-1}}(Y_{a'}) \leq G_{a'-1\,a'},$$

and the lemma follows. Thus we may assume that $[U_a, Y_{a'}] \neq 1$. In particular, $Y_{a'} \not\leq V_\beta$.

According to Lemma 10.6 (iii), $\widehat{U}_a \leq Q_{a'-1} \leq G_{a'}$. Hence

$$\widehat{U}_a = Z_a(\widehat{U}_a \cap Q_{a'}) .$$

Set $R = [Z_{\alpha'+1}, V_\beta \cap Q_{\alpha'}]$. Then $R \neq 1$ and, by Lemma 10.6 (i), $R \leq Y_{\alpha'}$. Since $Y_{\alpha'} \not\leq V_\beta$, $|R| = 2$ which implies that $V_\beta \cap Q_{\alpha'}$ centralizes a hyperplane, say X, of $Z_{\alpha'+1}$. So $[V_\beta : C_{V_\beta}(X)] \leq 2$ and thus, as $\eta(G_\beta, V_\beta) = 2$ by Lemma 10.4 (i), $X \leq Q_\beta$. Hence $X = Z_{\alpha'+1} \cap Q_\beta$. In particular, $[Z_{\alpha'+1} \cap Q_\beta, Z_\alpha \cap Q_{\alpha'}] = 1$ which gives that $Z_{\alpha'+1} \cap Q_\beta$ acts as a transvection on Z_α.

Observing that $[\widehat{U}_\alpha \cap Q_{\alpha'}, Z_{\alpha'+1} \cap Q_\beta] \leq Y_{\alpha'}$, it follows that either $R[\widehat{U}_\alpha \cap Q_{\alpha'}, Z_{\alpha'+1} \cap Q_\beta] = Y_{\alpha'}$ or $[\widehat{U}_\alpha \cap Q_{\alpha'}, Z_{\alpha'+1} \cap Q_\beta] = R$. The former case yields $[U_\alpha, Y_{\alpha'}] = 1$, so the latter must hold. As a consequence $Z_{\alpha'+1} \cap Q_\beta$ normalizes $Z_\alpha(\widehat{U}_\alpha \cap Q_{\alpha'}) = \widehat{U}_\alpha$. From Lemma 10.5 (ii) L_α/Q_α contains no transvections and therefore $\langle L_\alpha, Z_{\alpha'+1} \cap Q_\beta \rangle = G_\alpha$. Thus $U_\alpha = \widehat{U}_\alpha$, a contradiction since $[\widehat{U}_\alpha, Y_{\alpha'}] = 1$.

Finally, $U_\alpha \cap Q_{\alpha'}$ centralizes $Y_{\alpha'}$ (especially if $Y_{\alpha'} = Y_{\alpha'-2}$) and so $[U_\alpha \cap Q_{\alpha'}, V_{\alpha'}] \leq Y_{\alpha'}$.

LEMMA 10.8. *Suppose $b \geq 5$. Then for all $(\alpha, \alpha') \in \mathcal{C}$, $U_\alpha \not\leq G_{\alpha'}$.*

PROOF. Suppose the lemma is false. Then we have an $(\alpha, \alpha') \in \mathcal{C}$ with $U_\alpha \leq G_{\alpha'}$. Put $R = [V_\beta, V_{\alpha'}]$, and let $\Delta(\beta) = \{\alpha, \lambda, \alpha+2\}$.

Since $U_\alpha = V_\beta(U_\alpha \cap Q_{\alpha'})$ we also have $[U_\alpha, V_{\alpha'}] \leq Y_{\alpha'}R \leq Y_{\alpha'}[V_{\alpha'}, Q_{\alpha'-1}]$ by Lemma 10.7. Also, by the minimality of b, $U_{\alpha+2} \leq Q_{\alpha'-1}$ and hence $[U_{\alpha+2}, V_{\alpha'}] \leq [Q_{\alpha'-1}, V_{\alpha'}]$. Since $V_{\alpha'} \not\leq Q_\beta$ by Lemma 10.2 (ii), we see that $[W_\beta, V_{\alpha'}] \leq [Q_{\alpha'-1}, V_{\alpha'}]Y_{\alpha'}$. In view of Lemma 10.4 (ii), (iii) we have

(10.8.1) $|[W_\beta/V_\beta, V_{\alpha'}]| \leq 2$.

Because $V_{\alpha'} \not\leq Q_\beta$ there exists an involution $x \in V_{\alpha'}$ which interchanges α and λ. Hence, $|[U_\alpha U_\lambda/V_\beta, x]| \leq 2$ by (10.8.1). Since $V_\beta \leq U_\alpha \cap U_\lambda$ we then infer that $[U_\alpha : U_\alpha \cap U_\lambda] \leq 2$ and so, as G_β is 2-transitive on $\Delta(\beta)$, we have

(10.8.2) For $\gamma \in O(S_3)$ with $\Delta(\gamma) = \{\lambda_1, \lambda_2, \lambda_3\}$, $[U_{\lambda_i} : U_{\lambda_i} \cap U_{\lambda_j}] \leq 2$ for $i \neq j$, $i, j \in \{1, 2, 3\}$.

From the minimality of b, $[Z_\alpha, U_{\alpha'-3}] = 1$ and hence $[Z_\alpha, U_{\alpha'-3} \cap \cap U_{\alpha'-1}] = 1$. So $[U_{\alpha'-1} : C_{U_{\alpha'-1}}(Z_\alpha)] \leq 2$ by (10.8.2). Therefore, $[V_{\alpha'} : C_{V_{\alpha'}}(Z_\alpha)] \leq 2$ and so $\eta(G_{\alpha'}, V_{\alpha'}) \leq 1$, contradicting Lemma 10.4 (i). This completes the proof of Lemma 10.8. □

PROOF OF THE MAIN THEOREM. If $b \geq 5$, then combining Lemmas 10.7 and 10.8 gives $U_\alpha \not\leq G_{\alpha'-1\,\alpha'}$ and $Y_{\alpha'} = Y_{\alpha'-2}$ for all $(\alpha, \alpha') \in \mathcal{C}$. Thus for $(\alpha, \alpha') \in \mathcal{C}$, $U_\alpha \not\leq Q_{\alpha'-2}$, thence $(\alpha-2, \alpha'-2) \in \mathcal{C}$ for some $\alpha - 2 \in \Delta^{[2]}(\alpha)$.

And then $Y_{a'-4} = Y_{a'-2} = Y_{a'}$. Continuing in this fashion we obtain $Y_\beta = Y_{a'}$ which is against Lemma 10.6 (iv). Thus we have shown that $b \in \{1, 3\}$.

REFERENCES

[Ch1] A. CHERMAK, *Failure of factoristation modules and primitive amalgams of index* $(3,3)$, Comm. Algebra, **14**(4) (1986), pp. 591–665.

[Ch2] A. CHERMAK, *Large triangular amalgams whose rank-1 kernels are not all distinct*, Comm. Algebra, **14** (1986), pp. 667–706.

[Go] D. GOLDSCHMIDT, *Automorphisms of trivalent graphs*, Ann. of Math., **111** (1980), pp. 377-406.

[LPR1] W. LEMPKEN - C. PARKER - P. ROWLEY, (S_3, S_6)*-amalgams I, II, III*, Nova J. Algebra Geom., **3** (1995), no. 3, pp. 209–269; no. 4, pp. 271–311; no. 4, pp. 313–356.

[LPR2] W. LEMPKEN - C. PARKER - P. ROWLEY, *The structure of* (S_3, S_6)*-amalgams I, II*, Comm. in Algebra, **22** (11) (1994), pp. 4175-4215, **22** (11) (1994), pp. 4217–4301.

[ON] M. O'NAN, *A characterisation of the Rudvalis' group*, Comm. in Algebra (1978), pp. 107–147.

[PR] C. W. PARKER - P. J. ROWLEY, *Symplectic Amalgams*, Springer (2002).

Manoscritto pervenuto in redazione l'11 marzo 2003

An amalgam is an alloy of mercury with another metal. It may be a liquid, a soft paste or a solid, depending upon the proportion of mercury. These alloys are formed through metallic bonding, with the electrostatic attractive force of the conduction electrons working to bind all the positively charged metal ions together into a crystal lattice structure. Almost all metals can form amalgams with mercury, the notable exceptions being iron, platinum, tungsten, and tantalum. Silver-mercury amalgams are... Minelli - Rampampam. Katy Perry - I Kissed a Girl. Imany - Don't Be So Shy. MÃ¥neskin - I WANNA BE YOUR SLAVE. Billie Eilish - Happier Than Ever. Madcon - Beggin'. MÃ¥neskin - CORALINE. Queen - We Will Rock You. Ed Sheeran - Bad Habits. Doja Cat - Need to Know. Billie Eilish feat. Khalid - lovely. MÃ¥neskin - Zitti e Buoni. Lana Del Rey - Summertime Sadness. Tom Odell - Another Love. The Kid LAROI & Justin Bieber - Stay. Amalgam IV Discussion. Amalgam Tier continued Items/Weapons: Please note: Weapons and items include the skill and techniques needed to use them. Zoro's swords including Shuusui - 20 points Shuusui - 15 points Kabuto - 10 points Seastone Jutte - 10 points Seastone Handcuffs - 10 points Reject Dial - 10 points Rumble Balls (Set of 3) - 10 points Special Muggy Bombs (Set of 3) - 10 points Perfect Clima Tact - 10 points Kamikiri's Flame Dial Sword - 5 Points Funkfreed - 5 points Paulie's Rope - 5 points Ohm's Iron. Minelli - Rampampam. Katy Perry - I Kissed a Girl. Imany - Don't Be So Shy. MÃ¥neskin - I WANNA BE YOUR SLAVE. Billie Eilish - Happier Than Ever. Madcon - Beggin'. MÃ¥neskin - CORALINE. Queen - We Will Rock You. Ed Sheeran - Bad Habits. Doja Cat - Need to Know. Billie Eilish feat. Khalid - lovely. MÃ¥neskin - Zitti e Buoni. Lana Del Rey - Summertime Sadness. Tom Odell - Another Love. The Kid LAROI & Justin Bieber - Stay.