Continuum Mechanics: Basics and Notations

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Synonyms

mechanical field theory, tensor notation

Definition

Continuum mechanics is the branch of mechanics that seeks to describe the mechanical behaviour of bodies in terms of fields.

Overview

Continuum mechanics is a theory thats seeks to describe the mechanical behavior of bodies in terms of fields, presuming a continuous distribution of matter in space. It is classically divided into three parts, namely *kinematics*, *general balances*, and *material modeling*.

Kinematics is the geometry of deformable solids, concerned with defining appropriate deformation measures that are extracted, for example, from the displacement field of a body. These deformation measures enter the material modeling, which specify the stress tensor field for a given movement of a body. The latter has to satisfy the local balances of momentum and moment of momentum. Thus, the overall structure is

kinematics		material modeling		general balances
e.g. $\mathbf{E} = (\mathbf{u} \otimes \nabla_0 + \nabla_0 \otimes \mathbf{u})$	$\rightarrow$	e.g. $\mathbf{T} = \mathbb{C} \cdot\cdot \mathbf{E}$	$\rightarrow$	e.g. $\mathbf{T} \cdot \nabla + \rho \mathbf{b} = \mathbf{o}$

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where the second line is the specification of the first line for small strain linear quasistatic elasticity. While kinematics and general balances are material independent and apply universally, the individuality of the material is accounted for when connecting kinematics and balances through *material modeling*, which is a wide area. The large variety of materials in combination with the need for precise models has lead to the development of many specialized material modeling frameworks, like elasticity, plasticity, fracture mechanics, creep mechanics, some of which are covered by other articles in this book. However, kinematics and general balances are the same for all materials. On these we give a brief overview, and exemplify their interplay in the most common material modeling frameworks elasticity and plasticity. This yields second order partial differential equations for the displacement field, which give rise to initial and boundary value problems when calculating the deformations and stresses of a body.

Notation

The symbolic notation presented in the following is a compromise between different styles. The list of symbols below is in accordance with conventions applied in most books on the subject. Vectors are denoted as bold minuscules (like $\mathbf{a}$) and second-order tensors as bold majuscules (like $\mathbf{T}$ for the stresses, $\mathbf{E}$ for the strains, and $\mathbf{Q}$ and $\mathbf{R}$ for rotation tensors). The dyadic product and single scalar contractions are denoted like $(\mathbf{a}\otimes\mathbf{b}\otimes\mathbf{c})\cdot\cdot(\mathbf{d}\otimes\mathbf{e}) = (\mathbf{b}\cdot\mathbf{d})(\mathbf{c}\cdot\mathbf{e})\mathbf{a}$, with $\cdot$ being the usual scalar product between vectors. The association of the scalar products is such that an n-fold contraction of two tensors of order n inherits the positive definiteness of the scalar product between vectors,

$$\mathbf{A}\cdot\cdot\mathbf{A} = A_{ij}A_{kl}\mathbf{e}_i\otimes\mathbf{e}_j\cdot\cdot\mathbf{e}_k\otimes\mathbf{e}_l = A_{ij}A_{ij} = \|\mathbf{A}\|^2 \tag{1}$$

We sometimes omit the scalar dot when only vectors and second order tensors are involved, the meaning is generally clear from the context. The component representation is here exclusively w.r.t. cartesian coordinates with an orthonormal location-independent basis $\mathbf{e}_i$, e.g. $\mathbf{F} = F_{ij}\mathbf{e}_i\otimes\mathbf{e}_j$. The index 0 indicates the material description, i.e. it translates as "w.r.t. the material coordinates $\mathbf{x}_0$". The additional entries that appear when gradients are taken are added on the right, e.g. the deformation gradient is

$$\mathbf{F} = \mathbf{x}(\mathbf{x}_0)\otimes\nabla_0 = \partial x_i(x_{01},x_{02},x_{03})/\partial x_{0j}\mathbf{e}_i\otimes\mathbf{e}_j, \tag{2}$$

implying the nabla operator $\cdot\times\nabla_{(0)} = (\partial\cdot/\partial x_{(0)i})\times\mathbf{e}_i$, with $\times$ being a product that is defined for vectors. We have to apply the derivation only to the components, as the basis does not depend on $\mathbf{x}_0$. Transpositions of entries of multi-linear forms are denoted by an upper right T (the usual transposition) in case of bilinear forms, i.e. second order tensors. Higher order tensors require a more refined notation. For example, indices on the T may indicate which entries are transposed, different letters may indicate different transpositions (as in Itskov (2007)).

This allows for a compact and transparent notation, where the most important mathematical properties are captured in concise expressions. Nevertheless, other notations are used and may offer advantages in specific situations. Of course, the notation does not affect the mathematical properties, however transcribing between books may be confusing. The following major deviations exist:

- Sometimes left gradients are used, e.g. the gradient of a vector field is $\nabla_0 \otimes \mathbf{x}(\mathbf{x}_0)$. This is found often in the Russian literature, e.g. Lurie and Belyaev (2010)
- The scalar contractions may be carried out consecutively on outer vectors, e.g.

$$\mathbf{A} : \mathbf{B} = A_{ij} B_{kl}\, \mathbf{e}_i \otimes \mathbf{e}_j : \mathbf{e}_k \otimes \mathbf{e}_l = A_{ij} B_{ji}, \tag{3}$$

  see Altenbach (2015). Sometimes, the arrangement of the scalar dots (here : and $\cdot\cdot$) indicates the order in which the base vectors are multiplied (see, e.g. Sochi (2016)). The different scalar connections may be achieved also by transpositions prior to multiplication, e.g. $\mathbf{A} : \mathbf{B} = \mathbf{A} \cdot\cdot \mathbf{B}^T$.
- To avoid confusion with the scalar contractions, linear mappings are sometimes written with function brackets, like $\mathbf{A} = \mathbb{C}[\mathbf{B}]$ where we write $\mathbf{A} = \mathbb{C} \cdot\cdot \mathbf{B}$. Consequently, single scalar dots are then used to denote the inner products in any vector space, like $\alpha = \mathbf{A} \bullet \mathbf{B}$, where we write $\alpha = \mathbf{A} \cdot\cdot \mathbf{B}$, and scalar dots are left out when applying tensors successively, like $\mathbf{a} = \mathbf{A}\mathbf{B}\mathbf{v}$ (see for example Bertram (2012)).
- The distinction between the two placements is sometimes denoted by majuscules indicating vectors in the reference placement and minuscules indicating vectors in the current placement (e.g. Miehe (2003)).
- One must be careful when the dyadic product is applied to tensors. Here (and in most other books) it denotes basically a concatenation, $\mathbb{C} = \mathbf{A} \otimes \mathbf{B} = A_{ij} B_{kl} \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l$, but other conventions may apply (see e.g. Itskov (2007) Section 5.2).
- The dyadic product may well be omitted, e.g. $\mathbf{F} = \mathbf{x}(\mathbf{x}_0)\nabla_0$, which helps to get more compact expressions without loosing information (Altenbach 2015).
- Here we denote a tensor's order by upper and lower case font and style. A more general method is to indicate the order n by n over- or underscores. This notation is popular among french scholars, who summarize two bars to one tilde to shorten the expressions, see e.g. (Horák et al 2017).

List of symbols

The following table summarizes the most important definitions that are customary in continuum mechanics of solids. All densities are

Symbol	Meaning, dimension
A	Area, m^2
J	determinant of $\mathbf{F}$, dimensionless
l	stress power, $N/mm^2/s^1=W/mm^3$
M	mass, kg
V	Volume, m^3
w	elastic strain energy density, $J/m^3=\rho$ J/kg kg)=$\rho^{-1}N/m^2$
λ	plastic strain rate, dimensionless
λ_i	principle stretch/eigenvalue of $\mathbf{U}$ and $\mathbf{V}$/singular value of $\mathbf{F}$, dimensionless
ρ	Mass density, kg/m^3
σ	scalar stress, N/mm^2
∇	differential operator w.r.t. $\mathbf{x}$, 1/m
∇_0	differential operator w.r.t. $\mathbf{x}_0$, 1/m
$\mathbf{b}$	mass-specific force density, m/s^2
$\mathbf{d}$	moment of momentum, kg m^2/s
$\mathbf{n}$	normalized vector perpendicular to a surface, dimensionless
$\mathbf{p}$	momentum, kg m/s
$\mathbf{t}$	stress/traction vector, N/mm^2
$\mathbf{u}$	displacement vector, m
$\mathbf{u}_i$	eigenvector of the material stretch tensor $\mathbf{U}$, dimensionless
$\mathbf{v}_i$	eigenvector of the spatial stretch tensor $\mathbf{V}$, dimensionless
$\mathbf{x}$, $\mathbf{x}_0$	location or position vector, m
$\mathbf{B}$	left Cauchy-Green tensor, dimensionless
$\mathbf{C}$	right Cauchy-Green tensor, dimensionless
$\mathbf{D}$	symmetric part of $\mathbf{L}$, s^{-1}
$\mathbf{E}$	a strain tensor, dimensionless
$\mathbf{F}$	deformation gradient $\mathbf{x}(\mathbf{x}_0)\otimes\nabla_0$, dimensionless
$\mathbf{H}$	displacement gradient $\mathbf{u}(\mathbf{x}_0)\otimes\nabla_0$, dimensionless
$\mathbf{L}$	spatial velocity gradient $\dot{\mathbf{u}}\otimes\nabla$, s^{-1}
$\mathbf{Q}$, $\mathbf{R}$	orthogonal tensors/rotation tensors, dimensionless
$\mathbf{T}$	Cauchy stress tensor, a subscript labels other stress tensors, N/mm^2
$\mathbf{U}$	material stretch tensor, dimensionless
$\mathbf{V}$	spatial stretch tensor, dimensionless
$\mathbf{W}$	antisymmetric part of $\mathbf{L}$, s^{-1}

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Keywords Continuum mechanics â€¢ Tensor analysis â€¢ Indicial notation â€¢. Thermodynamics â€¢ Conservation principle â€¢ Constitutive models â€¢ Elasticity. 1 Introduction. Mechanics is the science that treats motions and forces, establishing the relations. between them. In brief, it is possible to imagine that a body is subjected to external.Â The study of continuum mechanics implies the use of tensor quantities and, because of that, it is important to have a background in tensor analysis. Continuum. From this basic foundation, continuum mechanics expands into equilibrium balances, constitutive models that relate material deformations to the stresses generated, and the 1st and 2nd Laws of Thermodynamics, which set limits on the behavior of the constitutive models. Upon completing these advanced topics, the Navier-Stokes equations can actually seem logical.Â Notation and Conventions. It is common in continuum mechanics to represent scalars with regular, normal-weight variables. For example, mass and entropy are represented by \(m\) and \(s\), respectively (although entropy is sometimes represented by \(\eta\)). Vectors, tensors, matrices, etc are represented by bolded variables such as \({\bf v}\) for velocity. A course on continuum mechanics intro- duces the basic principles of mechanics and prepares students for ad- vanced courses in traditional and emerging fields such as biomechanics and nanomechanics. This text introduces the main concepts of con- tinuum mechanics simply with rich supporting examples but does not compromise mathematically in providing the invariant form as well as component form of the basic equations and their applications to problems in elasticity, fluid mechanics, and heat transfer.Â The book features: derivations of the basic equations of mechanics in invariant (vector and tensor) form and specializations of the governing equations to various coordinate systems; numerous illustrative exam- ples; chapter-end summaries; and exercise...