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NOTE ON THE CORE MATRIX PARTIAL ORDERING

Jacek Mielniczuk

Department of Applied Mathematics and Computer Science
University of Life Sciences in Lublin
Akademicka 13, 20–950 Lublin, Poland

e-mail: jacek.mielniczuk@up.lublin.pl

Abstract

Complementing the work of Baksalary and Trenkler [2], we announce some results characterizing the core matrix partial ordering.

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1. Preliminaries

Let $\mathbb{C}^{m\times n}$ be the set of $m\times n$ matrices with complex entries. We will denote the conjugate transpose, range (column space), and nullspace of $A \in \mathbb{C}^{m\times n}$ by A^*, $R(A)$, and $N(A)$, respectively. P_A will stand for the orthogonal projector on $R(A)$. We use I to denote an identity matrix with dimensions following from the context.

We start by stating several basic facts on generalized inverses. As references, one can consult [4, Sections 2.2–2.5] or [5, Sections 4.2–4.5].

We let A^- designate a generalized inverse of A, this being defined as a solution to the matrix equation $AXA = A$. A least squares generalized inverse of $A \in \mathbb{C}^{m\times n}$, written as A_ℓ^-, is defined to be a solution to the matrix equation $AX = P_A$ ([4, Theorem 2.5.14]). The collection of all A_ℓ^- is denoted by $\{A_\ell^-\}$. In light of Theorems 2.5.24 (ii) and 2.5.27 in [4], the

Moore-Penrose inverse of A is the unique element A^+ of $\{A_\ell^-\}$ with the property $R(A^+) = R(A^*)$. The general expression of A_ℓ^- can be written as $A_\ell^- = A^+ + (I - A^+A)U$, where $U \in \mathbb{C}^{n\times m}$ is arbitrary ([4, Theorem 2.5.17]). We will use the following simple fact ([4, Theorem 2.5.28 (iv)]): $A^+ = (A^*A)^+ A^*$.

We shall mostly be concerned with core matrices. Recall that a square matrix A is said to be core if $R(A)$ and $N(A)$ are complementary subspaces, which is equivalent to saying that $R(A) = R\left(A^2\right)$. Given a core matrix A, we let Q_A represent the projector which projects a vector on $R(A)$ along $N(A)$. A c-inverse A_c^- of a core matrix A is defined to be a solution to the matrix equation $XA = Q_A$ ([4, Definition 6.4.1]). We let $\{A_c^-\}$ denote the collection of all A_c^-. Among the c-inverses, those having $R(A_c^-) = R(A)$ are called χ-inverses ([4, Definition 2.4.1]). According to Theorem 2.4.3 and Remark 2.4.14 of [4], the group inverse $A^\#$ is the uniquely determined χ-inverse satisfying the following condition $N\left(A^\#\right) = N(A)$. It is evident that $A^\#$ is a reflexive generalized inverse of A such that $AA^\# = A^\#A$ ([4, Theorem 2.4.6]).

Following [2], we define the core inverse $A^\oplus$ by $A^\oplus = A^\#AA^+$. In fact, $A^\oplus$ is the unique generalized inverse of A, which is both a least squares inverse and a χ-inverse of A. In [2] there are presented some results on characterizations of $A^\oplus$. Finally, let us point out that the core inverse coincides with the hybrid inverse $A_{\rho^*\chi}^-$ defined by Rao and Mitra [5, Section 4.10.2].

2. Core matrix partial order

We will be concerned here with the core relation defined by Baksalary and Trenkler [2].

Definition 1. For a pair of core matrices $A, B \in \mathbb{C}^{n\times n}$ we define the core relation $<^\oplus$ by saying that $A <^\oplus B$ if the following condition is satisfied:

$$A^\oplus(B - A) = (B - A)A^\oplus = 0. \tag{1}$$

The lemma below gives two other conditions that are equivalent to (1).

Lemma 2. *Let A and B be core matrices of the same order. Then the following statements are equivalent:*

1. $A <^{\oplus} B$,
2. $A^{+}(B-A) = (B-A)A^{\#} = 0$,
3. $A^{*}A = A^{*}B$ *and* $BA = A^{2}$.

Proof. We first recall the well-known fact ([3, Fact 2.10.12]) that $rank(AB) = rank(A)$ if and only if $R(AB) = R(A)$. This result implies, and is in fact equivalent to, the statement that $rank(AB) = rank(B)$ if and only if $N(AB) = N(B)$.

To establish the claim, observe that $A^{\oplus}$, A^{+}, $A^{\#}$ and A have the same rank. Hence, $R\left(A^{\oplus}\right) = R\left(A^{\#}\right) = R\left(A\right)$ and $N\left(A^{\oplus}\right) = N\left(A^{+}\right) = N\left(A^{*}\right)$, from which the required result follows. ■

Let us mention here another equivalent formulation of condition (1). As observed in [2, (3.21)], $A <^{\oplus} B$ if and only if $A^{+}B = A^{+}A$ and $BA = A^{2}$.

Another concept referred to is the minus partial ordering (see, for example, [4, Chapter 3]). We say that $A \in \mathbb{C}^{m\times n}$ is below $B \in \mathbb{C}^{m\times n}$ under the minus partial order, and write $A <^{-} B$, if $(A-B)A^{-} = 0$ and $A^{-}(A-B) = 0$ for some generalized inverse A^{-}.

It is worth making the following Proposition, which includes Theorem 8 in [2].

Proposition 3. *If* $A <^{\oplus} B$ *then* $A <^{-} B$, $R(A) \subset R(B)$, $R\left(A^{*}\right) \subset R\left(B^{*}\right)$. *The relation* $<^{\oplus}$ *is reflexive and antisymmetric.*

The following Theorem describes a new property of the core relation $<^{\oplus}$.

Theorem 4. $A <^{\oplus} B$ *if and only if* $\left\{B_{\ell}^{-}\right\} \subset \left\{A_{\ell}^{-}\right\}$ *and* $\left\{B_{c}^{-}\right\} \subset \left\{A_{c}^{-}\right\}$.

Proof. For proof of necessity, assume that $G \in \left\{B_{\ell}^{-}\right\}$. Since $A <^{\oplus} B$, we have $A^{*}A = A^{*}B$ and $R(A) \subset R(B)$. Therefore $A^{*}AG = A^{*}BB^{+} = A^{*}$. Premultiplying this relationship by $A\left(A^{*}A\right)^{+}$ yields $AG = AA^{+}$, which justifies $\left\{B_{\ell}^{-}\right\} \subset \left\{A_{\ell}^{-}\right\}$. Suppose next that $G \in \left\{B_{c}^{-}\right\}$. Since $BA = A^{2}$, we get $GA = GA^{2}A^{\#} = GBAA^{\#} = Q_{B}AA^{\#} = AA^{\#}$. This proves that $\left\{B_{c}^{-}\right\} \subset \left\{A_{c}^{-}\right\}$.

To show sufficiency, note that our assumption $\left\{B_{c}^{-}\right\} \subset \left\{A_{c}^{-}\right\}$ forces $A = B^{\#}A^{2}$. Then, clearly, $R(A) \subset R(B)$, and consequently, $BA = BB^{\#}A^{2} = A^{2}$, as needed. Next, to establish $A^{*}A = A^{*}B$, we consider the general expression $B_{\ell}^{-} = B^{+} + (I - B^{+}B)U$. If $\left\{B_{\ell}^{-}\right\} \subset \left\{A_{\ell}^{-}\right\}$, then $AB_{\ell}^{-} = AB^{+}$, and consequently, $A\,(I - B^{+}B)\,U = 0$ for every $U \in \mathbb{C}^{n\times n}$, which implies that $A = AB^{+}B$. Hence $R\,(A^{*}) \subset R\,(B^{*})$. Moreover, $\left\{B_{\ell}^{-}\right\} \subset \left\{A_{\ell}^{-}\right\}$ guarantees that $A^{*} = A^{*}AB^{+}$. Therefore $A^{*}B = A^{*}AB^{+}B = A^{*}A$, as required. ■

Theorem 4 guarantees that the core relation is transitive. On account of Proposition 3, we obtain that the relation $<^{\oplus}$ defines a matrix partial ordering ([2, Theorem 6]).

In the following we shall link different types of partial orders together. The following terminology will be required ([4, Definitions 6.3.1, 6.5.2]).

For $A, B \in \mathbb{C}^{m\times n}$, we define the left star relation $*<$ by saying that $A{*<}\,B$ if $R(A) \subset R(B)$ and $A^{*}A = A^{*}B$.

For core matrices $A, B \in \mathbb{C}^{n\times n}$ we define the right sharp relation $<\#$ by setting $A <\# B$ if $R\,(A^{*}) \subset R\,(B^{*})$ and $A^{2} = BA$.

The star relation is due to Baksalary and Mitra [1]. As is well known, the left star and the right sharp relation are partial orders ([1], [4, Corollary 6.3.10])

Proposition 3 permits us to conclude with the following

Proposition 5. *$A <^{\oplus} B$ if and only if $A{*<}\,B$ and $A <\# B$.*

As a matter of fact, Proposition 5 states that the core relation is an intersection partial ordering ([4, Definition A.8.1]).

Some remarks are due. It was our intention here to present a fairly simple and selfcontained proof of Theorem 4. However, once Proposition 5 is established, Theorem 4 may be achieved by appealing to characterizations of one-sided orders as given by Theorems 6.4.8 and 6.5.17 in [4].

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Partial dierential equations (PDEs) form the basis of very many math-ematical models of physical, chemical and biological phenomena, and more recently their use has spread into economics, nancial forecast-ing, image processing and other elds. To investigate the predictions of PDE models of such phenomena it is often necessary to approximate their solution numerically, commonly in combination with the analysis of simple special cases; while in some of the recent instances the numerical models play an almost independent role.Â On the other hand for supersonic ow where Mâˆž > 1, the system is equivalent to the one-dimensional wave equation and the system is hyperbolic. Part of our goal here is to clearly define and name three different chain rules and indicate in which situation they are appropriate. The chain rule is conceptually a divide and conquer strategy (like Quicksort) that breaks complicated expressions into sub-expressions whose derivatives are easier to compute.Â A word of caution about terminology on the web. Unfortunately, the chain rule given in this section, based upon the total derivative, is universally called â€œmultivariable chain ruleâ€ in calculus discussions, which is highly misleading! Only the intermediate variables are multivariate functions. Matrix partial orderings. 161. holds, then the first conditions in (b) and (es) are equivalent, which is partially related to the statement (b) ti (d),(e,) included in (2.1).Â Hartwig, R. E. 1978. A note on the partial ordering of positive semidefinite matrices, Linear and Multilinear Algelna 6223-226. Hartwig, R. E. 1980. How to partially order regular elements, Math. lapon.