Graph Laplacian and Lyapunov design of collective planar motions

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Abstract—In recent work, the authors have proposed a Lyapunov design to stabilize isolated relative equilibria in a kinematic model of identical all-to-all coupled particles moving in the plane at unit speed. This note presents an extension of these results to arbitrary connected topologies by considering a general family of quadratic Lyapunov functions induced by the Laplacian matrix of the communication graph.

1. Introduction

Feedback control laws that stabilize collective motions of particle groups have a number of engineering applications including unmanned sensor networks. For example, autonomous underwater vehicles (AUVs) are used to collect oceanographic measurements in formations that maximize the information intake, see e.g. [LPL$^+$05] and the references therein.

In this paper, we consider a kinematic model of identical (pointwise) particles in the plane [JK03]. The particles move at constant speed and are subject to steering controls that change their orientation. In recent work [SPL05], see also [SPL04, PLS05], we proposed a Lyapunov design to stabilize isolated relative equilibria of the model. Isolated relative equilibria either correspond to parallel motion of all particles with fixed relative spacing or to circular motion of all particles about a common center with fixed relative phases. The stabilizing feedbacks were derived from Lyapunov functions that prove exponential stability and suggest almost global convergence properties. The results in [SPL05] assume an all-to-all communication topology, that is, the feedback control applied to one given particle uses information about the (relative) heading and position of all other particles.

In the present note, we indicate how the all-to-all assumption can be relaxed to any bidirectional connected communication topology. We provide a unified interpretation of all the Lyapunov functions considered in earlier work as quadratic forms induced by the Laplacian of the graph associated to the communication topology.

The model assumptions are recalled in Section 2. Section 3 introduces the quadratic functions induced by the communication topology. The main Lyapunov functions considered in [SPL05] are then reinterpreted and generalized in Section 4. A short discussion concludes the paper in Section 6.

2. Particle model and control design

We consider a continuous-time kinematic model of $N > 1$ identical particles (of unit mass) moving in the plane at unit speed [JK03]:

$$\begin{aligned} \dot{r}_k &= e^{i\theta_k} \\ \dot{\theta}_k &= u_k, \end{aligned} \tag{1}$$

where $k = 1, \ldots, N$. In complex notation, the vector $r_k = x_k + iy_k \in \mathbb{C} \approx \mathbb{R}^2$ denotes the position of particle k and the angle $\theta_k \in S^1$ denotes the orientation of its (unit) velocity vector $e^{i\theta_k} = \cos\theta_k + i\sin\theta_k$. We use the boldface variable without index to denote the corresponding N-vector, e.g. $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_N)^T$. The configuration space consists of N copies of the group $SE(2)$. In the absence of steering control ($\dot{\theta}_k = 0$), each particle moves at unit speed in a fixed direction and its motion is decoupled from the other particles.

We study the design problem of choosing feedback controls that stabilize a prescribed collective motion. The feedback controls are identical for all the particles and only depend on relative orientation and relative spacing, i.e., on the variables $\theta_{kj} = \theta_k - \theta_j$ and $r_{kj} = r_k - r_j$, $j, k = 1, \ldots, N$. Consequently, the closed-loop vector field is invariant under an action of the symmetry group $SE(2)$ and the closed-loop dynamics evolve on a reduced quotient manifold (shape space). Equilibria of the reduced dynamics are called relative equilibria and can be only of two types [JK03]: *parallel* motions, characterized by a common orientation for all the particles (with arbitrary relative spacing), and *circular* motions, characterized by circular orbits of the particles around a fixed point.

The feedback control laws are further restricted by a limited communication topology. The communication topology is defined by a undirected graph $G(V,E)$ with N vertices in $V=\{1,\dots,N\}$ and e edges $(i,j)\in E$ whenever there exists a communication link between particle i and particle j. We note $\mathcal{N}(k)=\{j \mid (j,k)\in E\}$ the set of neighbors of k, that is, the set of vertices adjacent to vertex j. The control u_k is allowed to depend on r_{kj} and θ_{kj} only if $j\in\mathcal{N}(k)$.

3. Laplacian quadratic forms

Consider the (undirected) graph $G=(V,E)$ and let d_k be the degree of vertex k. The Laplacian L of the graph G is the matrix defined by

$$\begin{aligned} L_{k,j} &= d_k, \quad \text{if } k=j \\ &= -1, \quad \text{if } (k,j)\in E, \\ &= 0, \quad \text{otherwise} \end{aligned} \tag{2}$$

The Laplacian matrix plays a fundamental role in spectral graph theory [Chu97]. Only basic properties of the Laplacian are used in this paper. First, $L\mathbf{1}=0$, and the multiplicity of the zero eigenvalue is the number of connected components of the graph. As a consequence, the Laplacian matrix of a connected graph has one zero eigenvalue and $N-1$ strictly positive eigenvalues.

We denote by $<\cdot,\cdot>$ the standard inner product in $\mathbb{C}^N$. The quadratic form $Q(\boldsymbol{z})=<\boldsymbol{z},L\,\boldsymbol{z}>$ vanishes only when $\boldsymbol{z}=\mathbf{1}z_0$. It defines a norm on the shape space $\mathbb{C}^N/\mathbb{C}$ induced by the action of the group of rigid displacements $\boldsymbol{z}\rightarrow\boldsymbol{z}+\mathbf{1}z_0$.

Consider the valence matrix $D=\mathrm{diag}\{d_k\}$, the adjacency matrix A, and the incidence matrix $B\in\mathbb{R}^{N\times e}$ associated to the graph G. One easily shows that $L=D-A=BB^T$. Using the property $L=BB^T$, an alternative expression for the quadratic form $Q(\boldsymbol{z})$ is

$$Q(\boldsymbol{z})=\sum_{(k,j)\in E}|z_k-z_j|^2$$

In words, $Q(\boldsymbol{z})$ is thus the total length of the polygonal line connecting communicating vertices z_k. Two examples of communication topology will be considered in this note. The S^N topology corresponds to all-to-all communication. Its Laplacian matrix is $L=I_N-\frac{1}{N}\mathbf{1}\mathbf{1}^T$. It is a projector, that is, $L^2=L$. The quadratic form $Q(\boldsymbol{z})$ then takes the expression

$$Q(\boldsymbol{z})=\parallel L\boldsymbol{z}\parallel^2$$

which is the sum of the (squared) distances of vertices z_k, $1\le k\le N$, to their centroid $\frac{1}{N}\sum_{j=1}^N z_j$.

The D^N topology corresponds to a ring communication: each particle is connected to two other particles. The Laplacian is in this case the matrix

$$L=\begin{pmatrix} 2 & -1 & 0 & \dots & -1 \\ -1 & 2 & -1 & \dots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots & \\ -1 & 0 & \dots & -1 & 2 \end{pmatrix} \tag{3}$$

4. Phase synchronization and phase balancing

We consider the quadratic form $Q(\dot{\boldsymbol{r}})$ for the model (1), that is, when the N points $\dot{r}_k=e^{i\theta_k}$ lie on the unit circle. Its time-derivative along the solutions of (1) is

$$\dot{Q}=2\sum_{k=1}^N<i\dot{r}_k,L_k\dot{\boldsymbol{r}}>u_k \tag{4}$$

and the control

$$u_k=K<i\dot{r}_k,L_k\dot{\boldsymbol{r}}>=K<ie^{i\theta_k},L_ke^{i\boldsymbol{\theta}}>, \tag{5}$$

with $K\neq 0$, ensures that Q evolves monotonically along the closed-loop solutions since

$$\dot{Q}=2K\parallel\boldsymbol{u}\parallel^2=2K\parallel\frac{\partial Q}{\partial\theta}\parallel^2 .$$

Solutions of the closed-loop equation $\dot{\theta}=K\frac{\partial Q}{\partial\theta}$ must converge to the critical set of Q. Note that the control (5) only depends on the relative orientations of particle k with its neighbors $j\in\mathcal{N}(k)$.

Using the equality $L=D-A=BB^T$, one has several equivalent expressions for Q:

$$\begin{aligned} Q(\dot{\boldsymbol{r}}) &= <e^{i\boldsymbol{\theta}},Le^{i\boldsymbol{\theta}}> && (6)\\ &= \mathrm{trD}-<\mathrm{e}^{\mathrm{i}\boldsymbol{\theta}},\mathrm{Ae}^{\mathrm{i}\boldsymbol{\theta}}> && (7)\\ &= \frac{1}{2}\mathrm{trD}-\mathbf{1}_{\boldsymbol{e}}{}^{\mathrm{T}}\cos(\mathrm{B}^{\mathrm{T}}\boldsymbol{\theta}) && (8) \end{aligned}$$

Likewise, one has the equivalent expressions for the derivative

$$\frac{\partial Q}{\partial\theta}=2B\sin(B^T\boldsymbol{\theta}) \tag{9}$$

and

$$\frac{\partial Q}{\partial\theta_k}=2\sum_{j\in\mathcal{N}(k)}\sin(\theta_k-\theta_j) \tag{10}$$

The quadratic function $Q(\dot{\boldsymbol{r}})$ reaches its minimum when $\dot{\boldsymbol{r}}=\mathbf{1}e^{i\theta_0}$, that is, when all phases synchronize, which corresponds to a parallel motion. The control $u=-B\sin(B^T\boldsymbol{\theta})$ is proposed in [JMB04] to achieve synchronization in the phase model $\dot{\theta}=u$. It generalizes to arbitrary communication topologies the all-to-all sinusoidal coupling encountered in Kuramoto model [Kur84]. For the S^N topology (all-to-all communication), the quadratic function Q becomes

$$Q(\dot{\boldsymbol{r}})=\parallel Le^{i\boldsymbol{\theta}}\parallel^2=N^2(1-|\frac{1}{N}\sum_{k=1}^N e^{i\theta_k}|^2) \tag{11}$$

Up to a constant and a change of sign, it coincides with the phase potential $U(\theta) = |p_\theta|^2$ used in [SPL05], where $p_\theta = \frac{1}{N}\sum_{k=1}^{N} e^{i\theta_k}$ denotes the centroid of particles, or equivalently, its average linear momentum $\dot{R} = \frac{1}{N}\sum_{k=1}^{N} \dot{r}_k$. The parameter $|p_\theta|$ is a classical measure of synchrony of the phase variables $\boldsymbol{\theta}$ [Kur84, Str00]. It is maximal when all phases are synchronized (identical). It is minimal when the phases balance to result in a vanishing centroid. In the particle model (1), synchronization of the phases corresponds to a parallel formation: all particles move in the same direction. In contrast, balancing of the phases corresponds to collective motion around a fixed center of mass.

For the D^N topology (ring communication), the structure of the critical points of Q has been further investigated in [JPL05]. The matrix B is a square $N \times N$ matrix in this case and kerB $= \mathbf{1}$. Because critical points must satisfy $B\sin(B^T\boldsymbol{\theta}) = 0$, one concludes that all critical points of Q satisfy the phase locking condition

$$(k,j) \in E \Rightarrow \sin(\theta_k - \theta_j) = \alpha \tag{12}$$

for some constant α. Each critical point is thus characterized by a fixed angle ϕ_0 such that the phase difference between any two pair of connected particles is either ϕ_0 or $\pi - \phi_0$. The Hessian of Q at a critical point takes the simple expression

$$\frac{\partial^2 Q}{\partial\theta^2} = \cos\phi_0 B\Lambda_e B^T \tag{13}$$

where Λ_e is a diagonal matrix. Each diagonal element is $+1$ when the corresponding edge connects two points with phase difference ϕ_0 and -1 when the corresponding edge connects two points with phase difference $\pi - \phi_0$. As a consequence, all critical points of Q are saddles except when $\Lambda_e = \pm I_e$, in which case the phase locking condition (12) becomes

$$(k,j) \in E \Rightarrow \theta_k - \theta_j = \theta_0 \tag{14}$$

for a fixed angle $\theta_0 \in [0,\pi]$. It shown in [JPL05] that extrema of Q correspond to generalized regular polygons.

5. Stabilization of circular formations

Under the constant control $u_k = \omega_0$, $\omega_0 \neq 0$, the particle k rotates on a circle of radius $\rho_0 = 1/|\omega_0|$ centered at $c_k = r_k - i\rho_0 e^{i\theta_k}$. Achieving a circular formation amounts to synchronize all the particle centers. This prompts us to define $s_k = i\omega_0 c_k$ and to consider the quadratic function $Q(\boldsymbol{s})$ in analogy to what was done in the previous section. Note that $Q(\boldsymbol{s}) = Q(\dot{\boldsymbol{r}})$ in the limit when $\omega_0 = 0$.

The time-derivative of $Q(\boldsymbol{s})$ along the solutions of (1) is

$$\dot{Q} = 2\sum_{k=1}^{N} < i\dot{r}_k, L_k\boldsymbol{s} > (u_k - \omega_0) \tag{15}$$

and the control

$$u_k = \omega_0 + K < i\dot{r}_k, L_k\boldsymbol{s} >= \omega_0 + K < ie^{i\theta_k}, L_k\boldsymbol{s} > \tag{16}$$

ensures that Q evolves monotonically along the closed-loop solutions since $\dot{Q} = 2K \parallel \frac{\partial Q}{\partial\theta} \parallel^2$. Bounded solutions of the closed-loop system must converge to the critical set of Q. Note that the control (5) only depends on the relative orientations and relative positions of particle k with its neighbors $j \in \mathcal{N}(k)$.

The phase control (5) and the spacing control (16) can be combined as follows: the composite Lyapunov function

$$V(\boldsymbol{r},\boldsymbol{\theta}) = \kappa_1 Q(\boldsymbol{s}) - \kappa_2 Q(\dot{\boldsymbol{r}}), \quad \kappa_1 > 0 \tag{17}$$

is nonincreasing along the closed-loop solutions with the control

$$u_k = \omega_0 - \frac{\partial V}{\partial\theta_k} \tag{18}$$

It is of interest to rewrite the control (18) as

$$\begin{aligned} u_k \;\; &= \;\; \omega_0(1 - \kappa_1(d_k+1) < e^{i\theta_k}, \tilde{r}_k >) \\ &\quad + (\kappa_1 - \kappa_2)\sum_{j\in\mathcal{N}(k)} \sin(\theta_k - \theta_j) \end{aligned} \tag{19}$$

where

$$\tilde{r}_k = r_k - \frac{1}{d_k+1}(r_k + \sum_{j\in\mathcal{N}(k)} r_j) \tag{20}$$

is the relative distance from particle k to the centroid of particles connected to k. The first term in (19) is a spacing control that stabilizes rotation of particle k around this centroid. The second term in (19) is a phase control that stabilizes synchronized orientations when $\kappa_2 > \kappa_1$ and balanced orientations when $\kappa_2 < \kappa_1$. In an all-to-all communication, $\kappa_2 > \kappa_1$ causes an aggregated circular formation (all particles rotate on a circle in synchrony), whereas $\kappa_2 < \kappa_1$ stabilizes a balanced circular formation around a fixed center of mass.

The following result is proven in [SPL05] for the all-to-all communication topology, that is, when $L = I - \frac{1}{N}\mathbf{1}\mathbf{1}^{\boldsymbol{T}}$. The same proof works if L is the Laplacian of an arbitrary connected graph.

Theorem 1 *Consider the particle model (1) with the control (19). All solutions converge to a relative equilibrium defined by a circular formation of radius* $\rho_0 = |\omega_0|^{-1}$ *with direction determined by the sign of* ω_0*. If* $\kappa_1 \neq \kappa_2$*, the asymptotic phase arrangement is a critical point of* $Q(e^{i\boldsymbol{\theta}})$*.*

6. Discussion

Several variations of the above control laws are discussed in [SPL05] in order to stabilize isolated relative equilibria of the model (1). They are presented for an all-to-all communication topology but can be extended to arbitrary connected topologies using the results of this note. For instance, a particular phase arrangement of the vectors $\dot{r}_k$ can be stabilized provided it corresponds to a minimum of a phase potential $U(\theta)$. It is shown in [SPL05] that symmetric balanced patterns (symmetric arrangement of N phases consisting of M clusters uniformly spaced around the unit circle) can be stabilized with phase potentials of the form

$$U^{M,N} = \sum_{m=1}^{M} K_m |\frac{1}{N} \sum_{k=1}^{N} e^{im\theta_k}|^2 \qquad (21)$$

with $K_m > 0$ for $m = 1, \ldots, M-1$ and $K_M < 0$. The generalization of those phase potentials to arbitrary connected topologies is

$$Q^{M,N} = -\sum_{m=1}^{M} K_m Q(e^{im\boldsymbol{\theta}}) \qquad (22)$$

In this sense, the quadratic Lyapunov functions proposed in this paper can be combined in diverse ways and provide a versatile tool to design planar collectives.

An issue of interest is whether the results of the present note extend to time-varying and to unidirectional communication topologies. In a practical environment, the communication between different agents is typically limited by a given spatial range. Communication neighbors then coincide with spatial neighbors and might change as agents move. Time-varying and unidirectional topologies have been considered in the recent papers [AJM02, Mor05] for simplified models. Extension of these results to the model of this paper will be considered in a forthcoming publication.

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In the mathematical field of graph theory, the Laplacian matrix, also called the graph Laplacian, admittance matrix, Kirchhoff matrix or discrete Laplacian, is a matrix representation of a graph. The Laplacian matrix can be used to find many useful properties of a graph. Together with Kirchhoff's theorem, it can be used to calculate the number of spanning trees for a given graph. The sparsest cut of a graph can be approximated through the second smallest eigenvalue of its Laplacian by Cheeger's... @article{Paley2006CollectiveMO, title={Collective Motion of Self-Propelled Particles: Stabilizing Symmetric Formations on Closed Curves}, author={D. Paley and Naomi Ehrich Leonard and R. Sepulchre}, journal={Proceedings of the 45th IEEE Conference on Decision and Control}, year={2006}, pages={5067-5072} }.Â Graph Laplacian and Lyapunov design of collective planar motions. The Laplacian potential is further used here to construct Lyapunov functions that are suitable for the analysis of cooperative control systems on graphs. These Lyapunov functions depend on the graph topology, and based on them a Lyapunov analysis technique is introduced for cooperative multi-agent systems on graphs. Control protocols coming from such Lyapunov functions are distributed in form, depending only on information about the agent and its neighbors. Keywords. Laplacian Potential Lyapunov Analysis Cooperative Multi-agent Systems Communication Graph Topology Passive Nonlinear Systems. These keywords were added by machine and not by the authors. 1. INTRODUCTION. The Laplacian matrix of a graph and its eigenvalues can be used in several areas of mathematical research and have a physical interpretation in various physical and chemical theories. The related matrix â€" the adjacency matrix of a graph and its eigenvalues were much more investigated in the past than the Laplacian matrix. The reader is referred to the monographs [CDS, CDGT]. However, in the authorâ€™s opinion the Laplacian spectrum is much more natural and more important than the adjacency matrix spectrum. TL;DR â€" Given a Graph and its associated Laplacian (in context of Graph Convolution), the primary Eigen values gives intuition into graph structure such as connected components and Eigen vectorsâ€¦Â TL;DR â€" Given a Graph and its associated Laplacian (in context of Graph Convolution), the primary Eigen values gives intuition into graph structure such as connected components and Eigen vectors captures various spectrums in the graph pertaining to zero crossings. Prerequisite â€" https://medium.com/analytics-vidhya/graph-convolution-intuition-9416c0f51167 on Graph Convolution. In this post we will approach Graph Convolution as a â€œmultiplicationâ€ problem in Fourier domain (as discussed in the previous posts).