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### The EANSC: a weighted extension and axiomatization

Yu-hsien Liao

*Department of Applied Mathematics, National Pingtung University, Taiwan*

Ling-Yun Chung

*Graduate School of Technological and Vocational  
Education, YunTech*

Po-hang Wu

*Department of Applied Mathematics, National Pingtung  
University of Education*

### Abstract

Here we propose a weighted extension of the equal allocation of nonseparable costs (EANSC). Further, an axiomatization is also proposed by applying consistency.

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**Contact:** Yu-hsien Liao - [twincos@ms25.hinet.net](mailto:twincos@ms25.hinet.net), Ling-Yun Chung - [darlingyun@gmail.com](mailto:darlingyun@gmail.com), Po-hang Wu - [pulo6240@hotmail.com](mailto:pulo6240@hotmail.com)

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# 1 Introduction

The *equal allocation of nonseparable costs* (EANSC) is a well-known solution concept in cooperative game theory. Based on the EANSC, each player receives his or her marginal contribution plus an equal share of the remaining utility. Since all players are dissimilar, it is reasonable that weights could be assigned to the “players” to modify the discriminations among all players. In this note, we adopt the *weight functions* to propose a weighted extension of the EANSC, which we name as the *weighted allocation of nonseparable costs* (WANSC). Based on the WANSC, each player receives his or her marginal contribution plus the part of the remains that does not correspond to an equal share of the remains, but it corresponds to the given (relative) weights of each player.

*Consistency* is a crucial property of solutions. Consistency allows us to deduce, from the desirability of an outcome for some problem, the desirability of its restriction to each subgroup for the associated reduction the subgroup faces. It has been investigated in various classes of problems, always based on reductions. Moulin (1985) introduced a notion of consistency to characterize the EANSC. Hwang (2006) characterized the EANSC not by means of a consistency property with respect to a reduced game, but making use instead of a property that analyzes the behavior of the solution with respect to an associated game. Hwang and Hsiao (2007) showed that the EANSC can be expressed as the sum of both internal dividends and external losses. Subsequently, Liao (2008, 2010) extended the EANSC to multi-choice TU games and interval TU games respectively.

Here we study the reduction and related consistency introduced by Moulin (1985). Inspired by the axiomatization of the Shapley value (1953) due to Hart and Mas-Colell (1989), we also axiomatize the WANSC by means of the consistency and the *weighted two-person standardness*. The weighted two-person standardness asserts that all players allocate utilities by applying the weights proportionably in all two-person games.

## 2 The weighted allocation of nonseparable costs

Let  $U$  be a non-empty and finite set of players. A coalition is a non-empty subset of  $U$ . A coalitional game with transferable utility (TU game) is a pair  $(N, v)$  where  $N$  is a coalition and  $v$  is a mapping such that  $v : 2^N \rightarrow \mathbb{R}$  and  $v(\emptyset) = 0$ . Denote the class of all TU games by  $G$ . A *solution* on  $G$  is a function  $\psi$  which associates with each game

$(N, v) \in G$  an element  $\psi(N, v)$  of  $\mathbb{R}^N$ . Let  $\psi$  be a solution on  $G$ . We say that  $\psi$  satisfies **efficiency** if for all  $(N, v) \in G$ ,  $\sum_{i \in N} \psi_i(N, v) = v(N)$ .

**Definition 1** *The equal allocation of nonseparable costs (EANSC), denoted by  $\eta$ , is the solution which associates with  $(N, v) \in G$  and each player  $i \in N$  the value*

$$\eta_i(N, v) = \beta_i(N, v) + \frac{1}{|N|} \cdot [v(N) - \sum_{k \in N} \beta_k(N, v)], \quad (1)$$

where  $\beta_i(N, v) = [v(N) - v(N \setminus \{i\})]$ .

Next, we propose a weighted extension of the EANSC. Let  $w : U \rightarrow \mathbb{R}^+$  be a positive function, then  $w$  is called a **weight function**. Given  $(N, v) \in G$  and a weight function  $w$ , for all  $S \subseteq N$ , we define  $w(S) = \sum_{i \in S} w(i)$ . The weighted extension of the EANSC is as follows.

**Definition 2** *Let  $w$  be a weight function. The weighted allocation of nonseparable costs (WANSC),  $\eta^w$ , is the solution which associates with  $(N, v) \in G$  and all players  $i \in N$  the value*

$$\eta_i^w(N, v) = \beta_i(N, v) + \frac{w(i)}{w(N)} \cdot [v(N) - \sum_{k \in N} \beta_k(N, v)]. \quad (2)$$

### 3 Reduced game and axiomatization

In this section, we introduce the reduced game introduced by Moulin (1985) to characterize the WANSC by means of related consistency.

**Definition 3 (Moulin, 1985)** *Given a solution  $\psi$ ,  $(N, v) \in G$  and  $S \subseteq N$ , the reduced game  $(S, v_{S,\psi})$  with respect to  $\psi$  and  $S$  is defined by for all  $T \subseteq S$ ,*

$$v_{S,\psi}(T) = \begin{cases} 0 & , \text{ if } T = \emptyset, \\ v(T \cup (N \setminus S)) - \sum_{i \in N \setminus S} \psi_i(N, v) & , \text{ otherwise.} \end{cases}$$

The reduced game is based on the idea that, when renegotiating the solution  $\psi$  within  $S$ , it is assumed that the coalition  $T \subseteq S$  cooperates with all the members of  $N \setminus S$ , paying off each of them at initial payoff  $\psi_i(N, v)$ , where  $i \in N \setminus S$ . For the reduced game, there is a corresponding *consistency* as follows.

**Definition 4** *A solution  $\psi$  satisfies consistency if  $\psi_i(S, v_{S,\psi}) = \psi_i(N, v)$  for all  $(N, v) \in G$  with  $|N| \geq 2$ , for all  $S \subseteq N$  and for all  $i \in S$ .*

**Remark 1** Moulin (1985) characterized the EANSC by means of the consistency property.

**Lemma 1** The WANSC satisfies efficiency.

**Proof.** Let  $(N, v) \in G$ . By Definition 2,

$$\begin{aligned} \sum_{i \in N} \eta_i^w(N, v) &= \sum_{i \in N} \left[ \beta_i(N, v) + \frac{w(i)}{w(N)} \cdot \left[ v(N) - \sum_{k \in N} \beta_k(N, v) \right] \right] \\ &= \sum_{i \in N} \beta_i(N, v) + \sum_{i \in N} \frac{w(i)}{w(N)} \cdot \left[ v(N) - \sum_{k \in N} \beta_k(N, v) \right] \\ &= \sum_{i \in N} \beta_i(N, v) + \frac{w(N)}{w(N)} \cdot \left[ v(N) - \sum_{k \in N} \beta_k(N, v) \right] \\ &= v(N). \end{aligned}$$

Hence, the WANSC  $\eta^w$  satisfies efficiency. ■

**Lemma 2** The WANSC  $\eta^w$  satisfies consistency.

**Proof.** Given  $(N, v) \in G$  and  $S \subseteq N$ . If  $S = \{i\}$  for some  $i \in N$ , then by efficiency of  $\eta^w$ ,

$$\eta_i^w(S, v_{S, \eta^w}) = v_{S, \eta^w}(S) = v(N) - \sum_{k \neq i} \eta_k^w(N, v) = \eta_i^w(N, v).$$

Assume that  $|N| \geq 2$  and  $|S| \geq 2$ . By the definitions of  $\beta$  and  $v_{S, \eta^w}$ , for all  $i \in S$ ,

$$\begin{aligned} \beta_i(S, v_{S, \eta^w}) &= \left[ v_{S, \eta^w}(S) - v_{S, \eta^w}(S \setminus \{i\}) \right] \\ &= \left[ v(N) - \sum_{k \in N \setminus S} \eta_k^w(N, v) - v(N \setminus \{i\}) + \sum_{k \in N \setminus S} \eta_k^w(N, v) \right] \\ &= \left[ v(N) - v(N \setminus \{i\}) \right] \\ &= \beta_i(N, v). \end{aligned} \tag{3}$$

By equations (1), (2), (3) and Definition 3, for all  $i \in S$ ,

$$\begin{aligned} &\eta_i^w(S, v_{S, \eta^w}) \\ &= \beta_i(S, v_{S, \eta^w}) + \frac{w(i)}{w(S)} \cdot \left[ v_{S, \eta^w}(S) - \sum_{k \in S} \beta_k(S, v_{S, \eta^w}) \right] \text{ ( by equation (2) )} \\ &= \beta_i(N, v) + \frac{w(i)}{w(S)} \cdot \left[ v_{S, \eta^w}(S) - \sum_{k \in S} \beta_k(N, v) \right] \text{ ( by equation (3) )} \\ &= \beta_i(N, v) + \frac{w(i)}{w(S)} \cdot \left[ v(N) - \sum_{k \in N \setminus S} \eta_k^w(N, v) - \sum_{k \in S} \beta_k(N, v) \right] \text{ ( by Definition (3) )} \\ &= \beta_i(N, v) + \frac{w(i)}{w(S)} \cdot \left[ \sum_{k \in S} \eta_k^w(N, v) - \sum_{k \in S} \beta_k(N, v) \right] \text{ ( by efficiency of } \eta^w \text{ )} \\ &= \beta_i(N, v) + \frac{w(i)}{w(S)} \cdot \left[ \frac{w(S)}{w(N)} \cdot \left[ v(N) - \sum_{k \in N} \beta_k(N, v) \right] \right] \text{ ( by equation (2) )} \\ &= \beta_i(N, v) + \frac{w(i)}{w(N)} \cdot \left[ v(N) - \sum_{k \in N} \beta_k(N, v) \right] \\ &= \eta_i^w(N, v). \end{aligned}$$

Hence, the WANS C  $\eta^w$  satisfies consistency. ■

Inspired by Hart and Mas-Colell (1989), we characterize the WANS C by means of the properties of consistency and *weighted standard for two-person games*.

**Definition 5** *A solution  $\psi$  satisfies weighted standardness for two-person games (WST) if for all  $(N, v) \in G$  with  $N = \{i, j\}$ ,*

$$\psi_i(N, v) = [v(\{i, j\}) - v(\{j\})] + \left(\frac{w(i)}{w(i) + w(j)}\right) \cdot [v(\{j\}) + v(\{i\}) - v(\{i, j\})].$$

**Remark 2** *It is not difficult to derive that if a solution  $\psi$  satisfies WST and consistency, then for all  $(\{i\}, v) \in G$ ,  $\psi(\{i\}, v) = v(\{i\})$ . The technique of the proof can be found in Hart and Mas-Colell (1989).*

**Lemma 3** *Let  $\psi$  be a solution on  $G$ . If  $\psi$  satisfies WST and consistency, then it also satisfies efficiency.*

**Proof.** Suppose  $\psi$  satisfies WST and consistency. Let  $(N, v) \in G$ . If  $|N| = 1$ , then  $\psi$  satisfies efficiency by Remark 2. If  $|N| = 2$ , it is trivial that  $\psi$  satisfies efficiency by WST. Assume that  $|N| > 2$  and  $i \in N$ . By Remark 2 and definition of  $v_{\{i\}, \psi}$ ,

$$\psi_i(\{i\}, v_{\{i\}, \psi}) = v_{\{i\}, \psi}(\{i\}) = v(N) - \sum_{k \neq i} \psi_k(N, v). \quad (4)$$

By consistency of  $\psi$ ,

$$\psi_i(\{i\}, v_{\{i\}, \psi}) = \psi_i(N, v). \quad (5)$$

By equations (4) and (5),

$$v(N) = \sum_{k \in N} \psi_k(N, v).$$

Hence,  $\psi$  satisfies efficiency. ■

**Theorem 1** *A solution  $\psi$  on  $G$  satisfies WST and consistency if and only if  $\psi = \eta^w$ .*

**Proof.** By Lemma 2,  $\eta^w$  satisfies consistency. Clearly,  $\eta^w$  satisfies WST.

To prove uniqueness, suppose  $\psi$  satisfies WST and consistency. By Lemma 3,  $\psi$  satisfies efficiency. Let  $(N, v) \in G$ . Suppose  $|N| = 1$ . By

Remark 2 and Lemma 3,  $\psi(N, v) = \eta^w(N, v)$ . If  $|N| = 2$ , it is trivial that  $\psi(N, v) = \eta^w(N, v)$  by WST. The case  $|N| > 2$ : Let  $i \in N$  and  $S = \{i, j\}$  for some  $j \in N \setminus \{i\}$ . Then

$$\begin{aligned} \psi_i(N, v) - \eta_i^w(N, v) &= \psi_i(S, v_{S,\psi}) - \eta_i^w(S, v_{S,\eta^w}) \quad (\text{by consistency of } \psi, \eta^w) \\ &= \eta_i^w(S, v_{S,\psi}) - \eta_i^w(S, v_{S,\eta^w}) \quad (\text{by WST of } \psi, \eta^w) \\ &= \frac{w(i)}{w(i)+w(j)} \cdot \left[ v_{S,\psi}(S) + v_{S,\psi}(\{i\}) - v_{S,\psi}(\{j\}) \right] \\ &\quad - \frac{w(i)}{w(i)+w(j)} \cdot \left[ v_{S,\eta^w}(S) + v_{S,\eta^w}(\{i\}) - v_{S,\eta^w}(\{j\}) \right]. \end{aligned} \quad (6)$$

By the definitions of  $v_{S,\psi}$  and  $v_{S,\eta^w}$ ,

$$\begin{aligned} v_{S,\psi}(\{i\}) - v_{S,\psi}(\{j\}) &= \left[ v(N \setminus \{j\}) - v(N \setminus \{i\}) \right] \\ &= v_{S,\eta^w}(\{i\}) - v_{S,\eta^w}(\{j\}). \end{aligned} \quad (7)$$

By equations (6), (7), the definition of  $v_{S,\psi}$  and the efficiency of  $\psi$  and  $\eta^w$ ,

$$\begin{aligned} \psi_i(N, v) - \eta_i^w(N, v) &= \frac{w(i)}{w(i)+w(j)} \cdot \left[ v_{S,\psi}(S) - v_{S,\eta^w}(S) \right] \\ &= \frac{w(i)}{w(i)+w(j)} \cdot \left[ \psi_i(N, v) + \psi_j(N, v) - \eta_i^w(N, v) - \eta_j^w(N, v) \right]. \end{aligned}$$

That is, for all  $i, j \in N$ ,

$$\frac{w(j)}{w(i)+w(j)} \cdot \left[ \psi_i(N, v) - \eta_i^w(N, v) \right] = \frac{w(i)}{w(i)+w(j)} \cdot \left[ \psi_j(N, v) - \eta_j^w(N, v) \right].$$

By efficiency of  $\psi$  and  $\eta^w$ ,

$$0 = v(N) - v(N) = \sum_{j \in N} \left[ \psi_j(N, v) - \eta_j^w(N, v) \right] = \frac{w(N)}{w(i)} \cdot \left[ \psi_i(N, v) - \eta_i^w(N, v) \right].$$

Hence, for all  $i \in N$ ,  $\psi_i(N, v) = \eta_i^w(N, v)$ . ■

The following examples are to show that each of the axioms used in Theorem 1 is logically independent of the remaining axioms.

**Example 1** Define a solution  $\psi$  by for all  $(N, v) \in G$  and for all  $i \in N$ ,

$$\psi_i(N, v) = \frac{v(N)}{|N|}.$$

Clearly,  $\psi$  satisfies consistency, but it violates WST.

**Example 2** Define a solution  $\psi$  by for all  $(N, v) \in G$  and for all  $i \in N$ ,

$$\psi_i(N, v) = \begin{cases} \eta_i^w(N, v) & , \text{ if } |N| \leq 2, \\ \eta_i(N, v) & , \text{ otherwise.} \end{cases}$$

Clearly,  $\psi$  satisfies WST, but it violates consistency.

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Body weight, fat mass (FM), fat-free mass (FFM) and resting energy expenditure (REE) were measured throughout the study. Results: For the N=19 CON and N=17 INT who completed the intervention per protocol, weight loss was greater for INT ( $14.1 \pm 5.6$  vs  $9.1 \pm 2.9$  kg;  $P < 0.001$ ). INT had greater FM loss ( $12.3 \pm 4.8$  vs  $8.0 \pm 4.2$  kg;  $P < 0.01$ ), but FFM loss was similar (INT:  $1.8 \pm 1.6$  vs CON:  $1.2 \pm 2.5$  kg;  $P = 0.4$ ). Mean weight change during the 7 Å— 2-week INT energy balance blocks was minimal ( $0.0 \pm 0.3$  kg).Å Conclusions: Greater weight and fat loss was achieved with intermittent ER. Interrupting ER with energy balance 'rest periods' may reduce compensatory metabolic responses and, in turn, improve weight loss efficiency. 5 Minimal axiomatization. 6 Extensions. 7 Adding dynamic operators. 8 Concluding words.Å This axiomatization is then simplified and its various extensions are proposed in [22], which views the work on the logic of accident as a variation and continuation of the work done on contingency logic. Independently of the literature on the logic of accident, [23] provides a topological semantics for a logic of unknown truths and shows its completeness over the class of S4 models. As for a comprehensive treatment of accident logic, see [12].Å By applying the usual reduction method, we obtain a complete axiomatization for the dynamic extension of contingency and accident logic. Our contributions consist of the following Keywords: The EANSC; weighted extension; axiomatization; consistency. (search for similar items in EconPapers) JEL-codes: C7 C0 (search for similar items in EconPapers) Date: 2015-03-11 References: View references in EconPapers View complete reference list from CitEc Citations Track citations by RSS feed. Downloads: (external link) <http://www.accessecon.com/Pubs/EB/2015/Volume35/EB-15-V35-I1-P52.pdf> (application/pdf). Related works: This item may be available elsewhere in EconPapers: Search for items with the same title. Export reference: BibTeX RIS (EndNote, ProCite, RefMan) HTML/Text. Persistent link: <https://EconPapers.repec.org/RePEc:ebl:ecbull:eb-13-00560>. Access Statistics for this article. Abstract We give an axiomatization of the aspiration core on the domain of all TU-games using a relaxed feasibility condition, non-emptiness, individual rationality, and generalized versions of the reduced game property (consistency) and superadditivity. Our axioms also characterize the C-core ([Guesnerie and Oddou, 1979] and [Sun, Trockel, and Yang, 2008]) and the core on appropriate subdomains.