

# DIFFERENTIAL GEOMETRY AND MODERN COSMOLOGY WITH FRACTIONALLY DIFFERENTIATED LAGRANGIAN FUNCTION AND FRACTIONAL DECAYING FORCE TERM

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Some interesting aspects and features of geodesic equation with fractional decaying force term in Riemannian differential geometry and modern cosmology were discussed within the framework of fractionally differentiated Lagrangian function formulated recently by the author.

*Key words:* fractional action-like variational approach, Riemann geometry on manifolds, 4-dimensional Lorentzian manifolds, cosmology.

## 1. INTRODUCTION

It is well believed today that fractional calculus is a quite irreplaceable means for description and investigation of classical and quantum complex dynamical system with holonomic as well as with nonholonomic constraints [1]. The fractional derivatives and integrals describe more accurately the complex physical systems and at the same time, investigate more about simple dynamical systems. Fractional derivatives and integrals have recently been applied to many problems in physics, finance and hydrology, polymer physics, biophysics and thermodynamics, chaotic dynamics, chaotic advection, random Brownian walks, modeling dispersion and turbulence, viscoelastically damped structures, control theory, transfer equation in a medium with fractal geometry, stochasting modeling for ultraslow diffusion, kinetic theories, statistical mechanics, dynamics in complex media, wave propagation in complex and fractal media, astrophysics, cosmology, etc.<sup>1</sup> Dealing with fractional derivatives is not more complex than with usual differential operators. Today there exist many different forms of fractional integral operators, ranging from divided-difference types to infinite-sum types, including Grunwald-Letnikov fractional derivative, Caputo fractional derivative, etc. but the Riemann-Liouville Operator is still the most

<sup>1</sup> See [1] and references therein.

frequently used when fractional integration is performed. The study of fractional calculus opened new branches of thought and fills in the gaps of traditional standard calculus in ways that as of yet, no one completely assimilates or understands. Although various fields of application of fractional derivatives and integrals are already well done, some others have just started in particular the study of fractional problems of the Calculus of Variations (COV) and respective Euler-Lagrange type equations is a subject of current strong research and investigations. In 1996–97, F. Riewe used the COV with fractional derivatives and consequently obtained a version of the Euler-Lagrange equations (ELE) with fractional derivatives that combines the conservative and non-conservative cases [2]. In 2001–2002, another approach was developed by M. Klimek by considering fractional problems of the COV but with symmetric fractional derivatives and correspondent ELE's were obtained, using both Lagrangian and Hamiltonian formalisms [3]. In 2002, O. Agrawal extended Klimek problem and proved a formulation for variational problems with right and left fractional derivatives in the Riemann-Liouville sense [4]. In 2004 the ELE's of Agrawal were used by D. Baling and T. Avkar to investigate problems with Lagrangians which are linear on the velocities [5]. In all the above mentioned studies, ELE's depend on left and right fractional derivatives, even when the problem depend only on one type of them. In 2005, M. Klimek studied problems depending on symmetric derivatives for which ELE's include only the derivatives that appear in the formulation of the problem [6]. The major problem with all these approaches is the presence of non-local fractional differential operators and the adjoint of a fractional differential operator used to describe the dynamics is not the negative of itself. Other complicated problems arise during the mathematical manipulations as the appearance of a very complicated Leibniz rule (the derivative of product of functions) and the non-presence of any fractional analogue of the chain rule. In general, the physical reasons for the appearance of fractional equations are long-range dissipation and nonconservation. Recently, we proposed a novel approach known as fractional action-like variational approach (FALVA) to model nonconservative dynamical systems where fractional time integral introduces only one parameter  $\alpha$  while in other models an arbitrary number of fractional parameters (orders of derivatives) appear [7–11]. The derived Euler-Lagrange equations are similar to the standard one but with the presence of fractional generalized external force acting on the system. No fractional derivatives appear in the derived equations. The conjugate momentum, the Hamiltonian and the Hamilton's equations are shown to depend on the fractional order of integration  $\alpha$  and vary as inverse of time. In the present work, we will discuss within the same framework, some interesting fractional features of geometry on Riemannian and 4-dimensional Lorentzian manifolds and show the importance of the fractional variational approach to explain some interesting cosmological aspects

in agreement with astrophysical observations. The paper is organized as follows: in section II, we review the basic postulates of fractional functional action integral; in section III, we discuss a simple application: the problem of arc length, *i.e.* the shortest line between two points on a Riemann manifold with a positive definite metric which enables us to introduce the fractional geodesic equation. In section IV, we treat the fractional geodesics in Euclidean space and discuss some of its important features. In section V, we show the importance of fractional geodesic equations in Riemann geometry and in particular, in modern cosmology. Three epochs are discussed: the matter, the radiation and the inflation epochs. Finally, conclusions are given in section VI.

## 2. THE FRACTIONALLY DIFFERENTIATED LAGRANGIAN FUNCTION APPROACH

In this section, we review briefly two basic points of the fractional action-like variational “*classical*” approach in classical and field theory: the resulted Euler-Lagrange equations and the violation of Noether’s conservation theorems.<sup>2</sup> The Riemann-Liouville fractional integral is defined by:

$$({}_{t_0=0})I_{\tau}^{-\alpha}f(\tau) = \frac{1}{\Gamma(\alpha)} \int_{t_0=0}^t f(t)(t-\tau)^{\alpha-1} d\tau, \quad 0 < \alpha < 1. \quad (1)$$

Consider a smooth Riemann manifold  $M$  and let  $L$  be an admissible smooth Lagrangian function  $L: R \times TM \rightarrow R$ . Given a Riemann-Liouville fractionally differentiated Lagrangian function  $S_{0 < \alpha < 1}$  on the set of paths  $q(\tau)$ ,  $0 \leq \tau \leq t$  between two given points  $A = q(0)$  and  $B = q(t)$ , *i.e.* a function on the tangent bundle  $TM$ . For any piecewise smooth differentiable path  $q: [t_0, t_1] \rightarrow M$ , the fractionally differentiated Lagrangian function  $S_{0 < \alpha < 1}$  to  $L$  is defined by

$$S_{0 < \alpha < 1}[q] = \frac{1}{\Gamma(\alpha)} \int_{t_0=0}^t L(\dot{q}(\tau), q(\tau), \tau)(t-\tau)^{\alpha-1} d\tau = \int_{t_0=0}^t L(\dot{q}, q, \tau) dg_t(\tau), \quad (2)$$

$$[t_0, t_1] \in R$$

where  $L(\dot{q}, q, \tau)$  is the Lagrangian weighted with  $(t-\tau)^{\alpha-1}/\Gamma(\alpha)$  and  $\Gamma(1+\alpha)g_t(\tau) = t^\alpha - (t-\tau)^\alpha$  with the scaling properties  $g_{\mu t}(\mu\tau) = \mu^\alpha g_t(\tau)$ ,  $\mu > 0$ . In reality, we considered a smooth action integral (a time smeared

<sup>2</sup> The reader is referred to [7–11] and references therein for more details.

measure  $dg_t(\tau)$  on the time interval  $[0, t] \in R^+$  which can be rewritten as the strictly singular Riemann-Liouville type fractional derivative Lagrangian

$$S_{\beta \in (0,1)}[q] = D_t^{-1+\beta} L(\dot{q}(t), q(t), t) = \\ = \int_0^t L(\dot{q}(t), q(t), t) \frac{d\tau}{(t-\tau)^\beta} \xrightarrow{\beta \rightarrow 0} \int_0^t L(\dot{q}(t), q(t), t) d\tau,$$

and thereby retrieved the standard action integral or functional integral. In this work, we have  $\beta = 1 - \alpha$ ,  $\alpha \in (0, 1)$ . Such type of functionals is known in mathematical economy, describing, for instance, a so called “discounting” economical dynamics. The true fractional derivatives are also often, nowadays, used for describing so called “dissipative structures” appearing in nonlinear dynamical systems and etc. The problem now is to find the paths  $q(\tau)$  for which make  $S_{0 < \alpha < 1}$  stationary. For this, we consider a one-parameter family of paths  $q = q(\tau, \xi)$ ,  $0 \leq \tau \leq t$ ,  $-\beta \leq \xi \leq \beta$  of class  $C^2$  in  $(\tau, \xi)$  from  $A = q(0, \xi)$  to  $B = q(t, \xi)$  in agreement with a given path  $q = q(\tau)$  for  $\xi = 0$ . Consequently, the fractional action at the given path  $q(\xi, \tau)$  becomes a fractional function of  $\xi$  if  $q(\tau) = q(0, \tau)$  makes  $S_{0 < \alpha < 1}$  an extremum, that is  $(dS_{0 < \alpha < 1}/d\xi)_{\xi=0} = 0$  for all such one-parameter variations  $q(\xi, \tau)$  of  $q(\tau)$ . Let  $x = x(\tau, \xi)$  be the coordinate point of  $q(\tau, \xi)$ . Set  $L = L(x, \zeta)$  evaluated at  $\zeta = \dot{x}$  where  $x = (x^i)$  is a coordinate system on  $M$  and  $\zeta^i = dx^i$  are functions on tangent vectors. Then the initial curves  $x = x(\tau)$  satisfies the modified Euler-Lagrange equations (MEL)

$$\frac{\partial L}{\partial x^k} - \frac{d}{d\tau} \left( \frac{\partial L}{\partial \zeta^k} \right) = \frac{1-\alpha}{t-\tau} \frac{\partial L}{\partial \zeta^k} \equiv F_{\zeta^k, \alpha}, \quad (3)$$

**Proof.** We evaluate  $dS_{0 < \alpha < 1}/d\xi$  and we change the order of differentiation with respect to  $\tau$  and  $\xi$  and we integrate by part to find easily:

$$\frac{d[S_{0 < \alpha < 1}]}{d\xi} = \frac{1}{\Gamma(\alpha)} \left[ \frac{\partial L}{\partial \zeta^k} \frac{\partial x^k}{\partial \xi} \right]_{\tau=0}^{\tau=t} - \\ - \frac{1}{\Gamma(\alpha)} \int_{t_0=0}^t \left[ (t-\tau)^{\alpha-1} \frac{d}{d\tau} \left( \frac{\partial L}{\partial \zeta^k} \right) + (1-\alpha) \frac{\partial L}{\partial \zeta^k} (t-\tau)^{\alpha-2} - \frac{\partial L}{\partial x^k} (t-\tau)^{\alpha-1} \right] \frac{\partial x^k}{\partial \xi} d\tau$$

The first integral is zero because of the boundary conditions  $x(\xi, 0) = x(A)$  and  $x(\xi, t) = x(B)$ . In fact  $d[S_{0 < \alpha < 1}]/d\xi$  must vanish at  $\xi = 0$  for all  $x = x(t, \xi)$

satisfying the boundary conditions. By assuming  $x^k(\tau, \xi) = x^k(\tau) + \xi y^k(\tau)$ , then  $\partial x^k / \partial \xi$  at  $\xi = 0$  can be any  $C^1$  function  $y^k(\tau)$  vanishing at the boundary times. Thus  $\forall y^k(\tau)$

$$\frac{1}{\Gamma(\alpha)} \int_{t_0=0}^t \left[ (t-\tau)^{\alpha-1} \frac{d}{d\tau} \left( \frac{\partial L}{\partial \zeta^k} \right) + (1-\alpha) \frac{\partial L}{\partial \zeta^k} (t-\tau)^{\alpha-2} - \frac{\partial L}{\partial x^k} (t-\tau)^{\alpha-1} \right] y^k(\tau) d\tau = 0,$$

from which we deduce equations (3).

The new term ‘ $F$ ’ on the RHS of equations (3) can be interpreted as the fractional frictional force which are a common type of non conservative force. That is by treating the action as a fractional integral, a linear time-decreasing dissipative force term appears. When  $\alpha = 1$ , we fall into the standard action and the later fails to describe dissipative systems. The important benefit of the fractional action is that it imitates naturally the appearance of the time-dependent dissipative term in the dynamical equations without introducing any auxiliary coordinate in the Lagrangian or using Rayleigh dissipation function and especially without allowing the appearance the fractional derivatives in the Lagrangian and the Hamiltonian. If  $(x^i, \dot{x}^i)$  are holonomic local coordinates on  $TM$  such that  $\eta(\tau) = (x^i(\tau))$  and  $\dot{\eta}(\tau) = (\dot{x}^i(\tau))$ , then  $\eta$  is a solution of the fractional systems of nonlinear ordinary differential equations in one of the following forms:

$$\frac{\partial L}{\partial x^k}(\zeta^j, x^j, \tau) - \frac{d}{d\tau} \left( \frac{\partial L}{\partial \zeta^k} \right) (\zeta^j, x^j, \tau) = \frac{1-\alpha}{t-\tau} \frac{\partial L}{\partial \zeta^k}(\zeta^j, x^j, \tau), \quad (4)$$

$$\begin{aligned} \frac{\partial^2 L}{\partial \zeta^k \partial \zeta^l} \dot{\zeta}^l(\zeta^j, x^j, \tau) + \frac{\partial^2 L}{\partial \zeta^k \partial x^l} \zeta^l(\zeta^j, x^j, \tau) - \frac{\partial L}{\partial x^k}(\zeta^j, x^j, \tau) + \\ + \frac{\alpha-1}{\tau-t} \frac{\partial L}{\partial \zeta^k}(\zeta^j, x^j, \tau) = 0. \end{aligned} \quad (5)$$

$k = 1, \dots, n = \dim M$ . Dot denotes time derivative with respect to  $\tau$ .

We discuss now the energy problem within the framework of fractional action principle.

The energy  $E = E(p, v)$  associated to  $L = L(p, v)$  and defined in a coordinate system  $(x, \zeta)$  by:

$$E = \frac{\partial L}{\partial \zeta^k} \zeta^k - L. \quad (6)$$

is not a constant of motion.  $p$  and  $v$  are respectively the momentum and the velocity.

**Proof.** If  $x = x(\tau)$  satisfies the fractional Euler-Lagrange equations represented by (3), and we take  $\zeta = \dot{x}$ , then  $E = E(x, \zeta): TM \rightarrow R$  satisfies:

$$\begin{aligned} \frac{dE}{d\tau} &= \frac{d}{d\tau} \left( \frac{\partial L}{\partial \zeta^k} \dot{x}^k - L \right) = \frac{d}{d\tau} \left( \frac{\partial L}{\partial \zeta^k} \right) \dot{x}^k + \frac{\partial L}{\partial \zeta^k} \ddot{x}^k - \left( \frac{\partial L}{\partial x^k} \right) \dot{x}^k - \frac{\partial L}{\partial \zeta^k} \ddot{x}^k, \\ &= \left( \frac{d}{d\tau} \left( \frac{\partial L}{\partial \zeta^k} \right) - \left( \frac{\partial L}{\partial x^k} \right) \right) \dot{x}^k = \frac{1-\alpha}{\tau-t} \frac{\partial L}{\partial \zeta^k} \dot{x}^k. \end{aligned}$$

The Noether's symmetry theorems are violated. Conservation occurs when  $\alpha = 1$  or when  $\tau \rightarrow \infty$ . In other words, lets assume that  $\partial L / \partial x^k = 0$ . This effectively means that the Lagrangian has translation symmetry in the (generalized) coordinate  $x^k$  or in other words  $L(x^1, x^2, \dots, x^i + \Delta x^i \dots) = L(x^1, x^2, \dots, x^i \dots)$ . By fractional Euler-Lagrange equations (3), we find  $p^k \propto (\tau - t)^{1-\alpha}$  where  $p^k = \partial L / \partial x^k$ . Hence fractional Euler-Lagrange equations in classical mechanics implies that translational invariance of a generalized coordinate implies that the generalized momentum  $p^k$  is a not a constant of motion, or a conserved quantity. The fractionally differentiated Lagrangian function (1) is quasi-invariant under the infinitesimal  $\tilde{\epsilon}$ -parameter transformations

$$\bar{\tau} = \tau + \tilde{\epsilon} \kappa(\tau, q) + O(\tilde{\epsilon}^2), \quad (7)$$

$$\bar{q}(\bar{\tau}) = q(\tau) + \tilde{\epsilon} \omega(\tau, q) + O(\tilde{\epsilon}^2), \quad (8)$$

up to a gauge term  $\Lambda$  unless

$$\begin{aligned} L(\bar{\tau}, \bar{q}^k(\bar{\tau}), \bar{q}'^k(\bar{\tau})) (t - \bar{\tau})^{\alpha-1} \frac{d\bar{\tau}}{d\tau} &= \\ &= L(\tau, q^k(\tau), \dot{q}^k(\tau)) (t - \tau)^{\alpha-1} + \tilde{\epsilon} (t - \tau)^{\alpha-1} \frac{d\Lambda}{d\tau}(\tau, q^k(\tau), \dot{q}^k(\tau)) + O(\tilde{\epsilon}^2). \end{aligned} \quad (9)$$

If the fractionally differentiated Lagrangian function (2) is invariant up to a gauge term  $\Lambda$  with the fact that  $L(\tau, q^k(\tau), \dot{q}^k(\tau)) = -(\partial L(\tau, q^k(\tau), \dot{q}^k(\tau)) / \partial \dot{q}^k) \cdot (\omega - \dot{q}^k \kappa)$ , then

$$\begin{aligned} & \frac{\partial L(\tau, q^k(\tau), \dot{q}^k(\tau))}{\partial \dot{q}^k} \cdot \omega(\tau, q^k) + \\ & + \left[ L(\tau, q^k(\tau), \dot{q}^k(\tau)) - \frac{\partial L(\tau, q^k(\tau), \dot{q}^k(\tau))}{\partial \dot{q}^k} \cdot \dot{q}^k \right] \kappa(\tau, q^k) - \\ & - \Lambda(\tau, q^k(\tau), \dot{q}^k(\tau)), \end{aligned} \quad (10)$$

is a constant of motion. When the Lagrangian is not a function of  $q^k$ , the constant of motion took the form:

$$\frac{\partial L(\tau, q^k(\tau), \dot{q}^k(\tau))}{\partial \dot{q}^k} + (1-\alpha) \int \frac{\partial L(\tau, q^k(\tau), \dot{q}^k(\tau))}{\partial \dot{q}^k} \frac{1}{t-\tau} d\tau. \quad (11)$$

For  $\alpha = 1$ ,  $\partial L(\tau, q^k(\tau), \dot{q}^k(\tau)) / \partial \dot{q}^k$  is constant and  $L$  is conserved.

### 3. THE PROBLEM OF ARC LENGTH AND THE MODIFIED GEODESIC EQUATION

As a simple application and of particular interest simultaneously is the problem of arc length, *i.e.* the shortest line between two points on a Riemann manifold with a positive definite metric. In this case, we need to take  $L = \sqrt{P}$  which is a function on tangent vectors evaluated at the velocity vector  $\dot{q}(\tau)$  of  $q(\tau)$ . The corresponding quadratic form  $P$  is  $ds^2 = \sum g_{ij} dx^i dx^j$  where  $g_{ij} = g(\partial/\partial x^i, \partial/\partial x^j)$  [12]. In this way, the fractional action integral

$$S_{0<\alpha<1}[q] = \frac{1}{\Gamma(\alpha)} \int_{t_0=0}^t \sqrt{P} (t-\tau)^{\alpha-1} d\tau, \quad (12)$$

is independent of the parameterization of  $q(\tau)$ . Consequently,

$$\begin{aligned} \frac{d[S_{0<\alpha<1}]}{d\xi} &= \frac{1}{2\Gamma(\alpha)} \int_{t_0=0}^t \left( P^{-1/2} \frac{\partial P}{\partial \xi} \right)_{\xi=0} (t-\tau)^{\alpha-1} d\tau \rightarrow \\ &\rightarrow \frac{1}{2\Gamma(\alpha)} \int_{t_0=0}^t \left( \frac{\partial P}{\partial \xi} \right)_{\xi=0} (t-\tau)^{\alpha-1} d\tau, \end{aligned} \quad (13)$$

assuming  $P \equiv 1$  for  $\xi = 0$ . Thus, we can replace  $L = \sqrt{P}$  by  $L = P = g_{ij} \zeta^i \zeta^j$  and the speed  $\sqrt{P}$  according to theorem 2 is not constant. Let  $M$  be a smooth

Riemann manifold. A path  $q = q(\tau)$  makes the fractional action integral (12) stationary if and only if its parametric equation  $x^i = x^i(\tau)$  in any coordinate system  $(x^i)$  satisfies the equation:

$$\frac{1}{2} \frac{\partial g_{ij}}{\partial x^k} \frac{dx^i}{d\tau} \frac{dx^j}{d\tau} - \frac{d}{d\tau} \left( g_{ik} \frac{dx^i}{d\tau} \right) = \frac{1-\alpha}{t-\tau} \left( g_{ik} \frac{dx^i}{d\tau} \right). \quad (14)$$

Consequently, let  $U$  be an open subset of  $R^n$ . Given a path  $\gamma: [t_1, t_2] \rightarrow U$  where we may define the fractional length of  $\gamma$  as (2). Then  $\gamma^k$  satisfies the differential equation

$$\frac{d^2 \gamma^k}{d\tau^2} + \frac{\alpha-1}{\tau-t} \frac{d\gamma^k}{d\tau} + \sum_{i,j} \frac{d\gamma^i}{d\tau} \cdot \Gamma_{ij}^k \cdot \frac{d\gamma^j}{d\tau} = 0, \quad (15)$$

where

$$\Gamma_{ij}^k = \frac{1}{2} \sum_r (\partial_i g_{jr} + \partial_j g_{ir} - \partial_r g_{ij}) \cdot g^{rk}, \quad (16)$$

is the Christoffel symbol. The one for geodesic motion is

$$\frac{d^2 x^k}{d\tau^2} + \frac{\alpha-1}{\tau-t} \frac{dx^k}{d\tau} + \Gamma_{ij}^k \frac{dx^i}{d\tau} \frac{dx^j}{d\tau} = 0. \quad (17)$$

Written as a system of first order equations, the integral curves are:

$$\frac{dx^k}{d\tau} = v^k, \quad (18)$$

$$\frac{dv^k}{d\tau} + \frac{\alpha-1}{\tau-t} v^k + \Gamma_{ij}^k v^i v^j = 0. \quad (19)$$

Equation (19) can be written in fact as:

$$\frac{dv^k}{d\tau} = -\Gamma_{ij}^k v^i v^j + \frac{1-\alpha}{\tau-t} v^k \equiv -\Gamma_{ij}^k v^i v^j + F^k, \quad (20)$$

where

$$F^k \equiv \frac{1-\alpha}{\tau-t} v^k, \quad (21)$$

is the fractional decaying forcing term. From a control theory point of view,  $F^k$  is the corresponding input weak decaying vector field [13]. In fact, by defining  $\dot{x}^\sigma \equiv dx^\sigma/dT = y^\sigma$ , equation (15) is identical to a Langevin equation with a time-dependent friction term in case a random source characterizing the properties of medium where motion occurs is applied (for example a random or stochastic electromagnetic field).



#### 4. THE MODIFIED GEODESICS IN EUCLIDEAN SPACE

As an example, consider the geodesics in Euclidean space with Euclidean metric

$$ds^2 = (dx^1)^2 + \dots + (dx^n)^2. \quad (22)$$

In this way, equations (14) read

$$\frac{d^2 x^i}{d\tau^2} = \frac{1-\alpha}{\tau-t} \frac{dx^i}{d\tau}, \quad (23)$$

yielding  $x^i(\tau) = c^i (\tau-t)^{2-\alpha} + x_0^i$ ,  $0 < \alpha < 1$ . So the geodesics are not straight lines as in the standard case. It is clear that they traverse at an accelerating velocity for  $0 < \alpha < 1$ .

Another interesting example concerns a sphere of unit radius in  $R^3$  with metric given simply by

$$ds^2 = d\theta^2 + \sin^2 \theta d\phi^2. \quad (24)$$

The geodesics equations are the extremum of the fractional like-energy:

$$\begin{aligned} E[\Lambda] &= \frac{1}{\Gamma(\alpha)} \int_{t_0=0}^t L(\theta, \phi, \dot{\theta}, \dot{\phi})(t-\tau)^{\alpha-1} d\tau = \\ &= \frac{1}{\Gamma(\alpha)} \int_{t_0=0}^t \frac{1}{2} (\dot{\theta}^2 + \cos^2 \theta \dot{\phi}^2) (t-\tau)^{\alpha-1} d\tau. \end{aligned} \quad (25)$$

The corresponding extremum are defined by the modified Euler-Lagrange equations (3) and we obtain:

$$\ddot{\theta} + \frac{\alpha-1}{T} \dot{\theta} = \frac{1}{T^{\alpha-1}} \frac{d}{dT} (T^{\alpha-1} \dot{\theta}) = -\cos \theta \sin \theta \dot{\phi}^2, \quad (26)$$

$$\frac{1}{T^{\alpha-1}} \frac{d}{dT} (T^{\alpha-1} \cos^2 \theta \dot{\phi}) = 0. \quad (27)$$

Equation (27) yields  $\cos^2 \theta \dot{\phi} = (T/T_0)^{\alpha-1}$ ,  $T_0$  is a constant of integration consequently, the differential equation (21) became:

$$\ddot{\theta} + \frac{\alpha-1}{T} \dot{\theta} + \tan \theta (1 + \tan^2 \theta) \left( \frac{T_0}{T} \right)^{2(1-\alpha)} = 0. \quad (28)$$

Note that for  $\theta(T) = 0$ ,  $\dot{\phi} = (T/T_0)^{\alpha-1}$  or  $\phi \propto T^\alpha$ . If  $0 < \alpha < 1$ , than  $\phi$  increases slowly with time. In reality, the modified equations of motion of free

test particles or equivalently, the geodesic equations can be easily determined within the context of ordinary vector analysis [14]. In fact, we choose rectangular Galilean coordinates  $(x_i, \tau)$ ,  $i = 1, 2, 3$  with the orthonormal spatial basis  $(\hat{e}_i)$ . The modified Euler-Lagrange equations are:

$$\frac{\partial L}{\partial \vec{r}} - \frac{d}{d\tau} \left( \frac{\partial L}{\partial \vec{v}} \right) = \frac{1-\alpha}{t-\tau} \frac{\partial L}{\partial \vec{v}}, \quad (29)$$

where  $L = L(\vec{r}, \vec{v}, \tau) = c^{-1} (c^2 - \vec{u} \cdot \vec{u})^{1/2}$  is the Lagrangian,  $\vec{u} \equiv \vec{v} - \vec{w}$  is the absolute velocity of the clock relative to physical space,  $c$  is the speed of light,  $\vec{v} = d\vec{r}/d\tau$  is the coordinate velocity of the clock and  $\vec{w} = \vec{w}(\vec{r}, \tau)$  is the flow velocity. It is easy to verify that  $\partial L / \partial \vec{r} = L^{-1} c^{-2} (\nabla \vec{w}) \cdot \vec{u}$  and  $\partial L / \partial \vec{v} = -L^{-1} c^{-2} \vec{u}$ . Consequently, equations (29) give the modified geodesic equation in Galilean coordinates as it is expected from the previous arguments:

$$\frac{d\vec{u}}{d\tau} + \frac{\alpha-1}{\tau-t} \vec{u} + \left( \bar{\mathbb{I}}_3 - \frac{\vec{u}\vec{u}}{c^2} \right) \cdot \nabla \vec{w} \cdot \vec{u} = 0, \quad 0 \leq u < c, \quad (30)$$

where  $\bar{\mathbb{I}}_3$  is the identity tensor in 3-space and  $\bar{\mathbb{I}}_3 - (\vec{u}\vec{u}/c^2)$  is the symmetric tensor of motion. The non-relativistic approximation of equation (25) is:

$$\frac{d\vec{u}}{d\tau} + \frac{\alpha-1}{\tau-t} \vec{u} + \nabla \vec{w} \cdot \vec{u} \equiv \frac{d\vec{v}}{d\tau} - \frac{d\vec{w}}{d\tau} + \frac{\alpha-1}{\tau-t} (\vec{v} - \vec{w}) + \nabla \vec{w} \cdot (\vec{v} - \vec{w}) = 0. \quad (31)$$

Simple mathematical manipulations lead to:

$$\frac{d\vec{v}}{d\tau} + \frac{\alpha-1}{\tau-t} \vec{v} = \frac{\partial \vec{w}}{\partial \tau} + \frac{\alpha-1}{\tau-t} \vec{w} + \frac{1}{2} \nabla w^2 + (\nabla \times \vec{w}) \times \vec{v}. \quad (32)$$

This is to say that the Newtonian force acting on a free test body or a satellite of mass  $m$  is:

$$\begin{aligned} \vec{F}_{\text{satellite}} &= m \vec{a}_{\text{satellite}} \equiv m \frac{d\vec{v}}{d\tau} = \\ &= m \frac{\partial \vec{w}}{\partial \tau} + \frac{\alpha-1}{\tau-t} m (\vec{w} - \vec{v}) + \frac{1}{2} m (\nabla w^2) + m ((\nabla \times \vec{w}) \times \vec{v}). \end{aligned} \quad (33)$$

The first term on the RHS of equation (33) represents the accelerating force arising from an explicit dependence of the flow on time; the second term represents the fractional weak decaying friction force; the third term represents the gravitational and generalized centrifugal accelerating force and the last term the generalized Coriolis accelerating force.

This equation is important when we deal with the problem of weak dissipative motion of test particles or bodies in rotating and accelerated frames of reference in Newtonian mechanics.

### 5. THE ROLE OF THE MODIFIED GEODESIC EQUATIONS IN MODERN MATHEMATICAL COSMOLOGY

In return to differential geometry on manifolds, we would like to review, clarify and critically analyze about the role of the modified geodesic equations in modern mathematical cosmology, in particular the Friedmann-Robertson-Walker (FRW) model of general relativistic cosmology. Geometrically, a FRW isotropic and homogeneous Riemannian manifold model is a 4-dimensional Lorentzian manifold  $M$  which can be expressed as a warped product  $I \times_R \Sigma$ , where  $I$  is an open interval of the pseudo-Euclidean manifold  $R_1^1$ ,  $\Sigma$  is a complete and connected 3-dimensional Riemannian manifold and the warping function  $R$  is a smooth, real-valued and non-negative function upon  $I$  [15]. The Lorentzian metric on  $M$  is written as:

$$g \equiv ds \otimes ds = -d\tau \otimes d\tau + a^2(\tau)\Omega, \quad (34)$$

where  $a(\tau)$  is the scale factor and which determines the time evolution of the spatial geometry of the universe and  $\Omega$  is the metric tensor on  $\Sigma$  which is considered as globally isotropic. In particular, every homogeneous Riemannian manifold  $(\Sigma, \Omega)$  is diffeomorphic to some 3-dimensional Lie group. It is easy to prove that the modified geodesic equations reduce to [8]:

$$div \bar{\gamma} + \frac{\alpha-1}{\tau-t} \partial_i v^i + c^2 R_{00} = 0, \quad (35)$$

$c$  being the celerity of light (in natural unit),  $v^i = dx^i/d\tau$  the particle velocity and  $div \bar{\gamma} = \partial_i \gamma^i = -4\pi G\rho$  being the Poisson equation,  $\rho$  the mass density in gravitational interaction,  $G$  is Newton gravitational constant and  $\gamma$  the acceleration. It follows that  $R_{00} = 4\pi\rho G_{eff}/c^2$  where

$$G_{eff} = G \left( 1 + \frac{1-\alpha}{4\pi\rho GT} \partial_i v^i \right) \equiv G + \Delta G, \quad (36)$$

is the effective Newton gravitational constant.<sup>3</sup> In order now to take into consideration the spacetime expansion, we will make use of the Raychaudhuri expansion scalar factor  $\dot{\theta}$  defined by [16]:

<sup>3</sup> In fact, the time-component (00) of the Ricci scalar curvature is  $R_{00} \approx \partial_i \Gamma_{00}^i$  [15]. Here  $R_{ab} = g^{ij} R_{ajbi}$  be the Ricci tensor where

$$R_{abcd} = \frac{1}{2} (g_{bc,ad} - g_{bd,ac} + g_{ad,bc} - g_{ac,bd}) + \Gamma_{ad}^j \Gamma_{bjc} - \Gamma_{ac}^j \Gamma_{bjd},$$

is the covariant curvature tensor;  $\Gamma_{ijk} = g_{pj} \Gamma_{ik}^p$ .

$$\partial_i v^i \equiv 3\dot{\theta} = 3\dot{a}/a. \quad (37)$$

The perturbed gravity reduces to

$$\Delta G = \frac{3(1-\alpha)\dot{a}}{4\pi\rho T a}. \quad (38)$$

The cosmological models are usually described by the Einstein field equations ( $c = 1$ ):

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -8\pi GT_{\mu\nu}, \quad (39)$$

where  $R = g^{ab}R_{ab}$  is the Ricci scalar curvature and  $T_{\mu\nu} = (p + \rho)u_\mu \otimes u_\nu + pg_{\mu\nu}$  is the stress-energy tensor of a perfect fluid with pressure  $p$  and density  $\rho$  which are both scalar fields on  $M$  and are constants on each hypersurface  $\Sigma_\tau \equiv \Sigma_0 = \tau \times \Sigma$  but which exhibit time-dependence. In what follows, we will assume that the constant sectional curvature of the 3-dimensional Riemannian space from  $(\Sigma, \Omega)$  is zero. That is the manifold is too flat. We will discuss in what follows three possibilities, *i.e.* three epochs:

**1. The Matter Epoch.** In the absence of the cosmological constant, the so-called modified Friedmann equations for dust ( $p = 0$ ) are:

$$\frac{\dot{a}^2}{a^2} + \frac{2(\alpha-1)\dot{a}}{T a} = \frac{8\pi G\rho}{3}, \quad (40)$$

$$\frac{\ddot{a}}{a} + \frac{\alpha-1}{T} \frac{\dot{a}}{a} = -\frac{4\pi G\rho}{3}, \quad (41)$$

with  $T = \tau - t$  is a new time change variable. Notice that equation (40) can be written as

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G\rho}{3} + \frac{\bar{\Lambda}_{decaying}}{3}. \quad (42)$$

$\bar{\Lambda}_{decaying} \equiv 6(1-\alpha)\dot{a}/aT$  can be viewed as a decaying cosmological constant with cosmic time.

This dissipative term can be introduced in the same way as in decaying two-fluid hydrodynamics and obeys the conservative law:  $T_{\mu\nu}^{matter} + T_{\mu\nu}^{curvature} + T_{\mu\nu}^{\bar{\Lambda}_{decaying}} = 0$  where  $T_{\mu\nu}^{curvature} = -(1/8\pi G)G_{\mu\nu}$ ,  $G_{\mu\nu} = R_{\mu\nu} - (1/2)g_{\mu\nu}R$  is the Einstein curvature tensor. The new decaying term does not influence the matter-conservation law  $T_{\mu;\nu}^{(matter)} = 0$ . One of the motivations for introducing a decaying lambda term is to reconcile the age parameter and the density parameter

of the universe with current observational data. This is to say that within the context of fractional action integral, the vacuum energy in Einstein field theory is replaced by external weak decaying force. For a dust-epoch manifold, *i.e.*  $p = 0$ , the dynamical equations within the framework of fractional action integral (geodesic equation with input weak decaying vector field) reduce to:

$$\frac{\dot{a}^2}{a^2} + 2\frac{\ddot{a}}{a} + \frac{4(\alpha-1)}{T}\frac{\dot{a}}{a} = 0, \quad (43)$$

and consequently, the scale factor reduces to  $a(T) \propto T^{(6-4\alpha)/3}$  and corresponds for  $\alpha < 3/4$  to an accelerating manifold expansion [17, 18]. There will be no need to invoke any types of exotic matter in order to accelerate the universe; all what we need is a weak decaying forcing term added to the geodesic equation and to treat the universe as a simple mechanical control system. For a dust-manifold, it is an easy task to show that the material density decreases with time as

$$\rho_{p=0} = \frac{(3-2\alpha)(2-\alpha)}{2\pi GT^2}, \quad (44)$$

and is positive for  $\alpha < 3/4$ .

**2. The Radiation Epoch.** For a radiation-epoch manifold, *i.e.*  $p = \rho/3$ , the dynamical equations within the framework of fractional action integral reduces to:

$$\frac{\dot{a}^2}{a^2} + \frac{2(\alpha-1)}{T}\frac{\dot{a}}{a} = \frac{8\pi G\rho}{3}, \quad (45)$$

$$\frac{\ddot{a}}{a} + \frac{2(\alpha-1)}{T}\frac{\dot{a}}{a} = -\frac{8\pi G\rho}{3}, \quad (46)$$

and consequently, the scale factor reduces to  $a(T) \propto T^{(5-4\alpha)/2}$  and corresponds again for  $\alpha < 3/4$  to an accelerating manifold expansion. Consequently, the material density decreases with time as

$$\rho_{p=\rho/3} = \frac{3(5-4\alpha)(3-2\alpha)}{16\pi GT^2}, \quad (47)$$

and is positive for  $\alpha < 3/4$ .

**3. The Vacuum Case or Inflation.** For a vacuum manifold, *i.e.*  $p = -\rho$ , the dynamical equations reduce to:

$$\frac{\dot{a}^2}{a^2} + \frac{2(\alpha-1)}{T}\frac{\dot{a}}{a} = \frac{8\pi G\rho}{3}, \quad (48)$$

$$\frac{\ddot{a}}{a} + \frac{2(\alpha-1)\dot{a}}{T} = \frac{8\pi G\rho}{3}, \quad (49)$$

and consequently, the scale factor reduces to  $a(T) \propto \exp(mT)$ ,  $m$  is a constant parameter and corresponds to an inflationary manifold for  $m > 0$  and deflationary manifold for  $m < 0$  [17].

The new thing in this scenario is that it is independent on whether the density is constant or variable. The material density varies with time as

$$\rho = \frac{3e^{mT}}{8\pi G} \left( e^{mT} + \frac{2(\alpha-1)}{T} \right). \quad (51)$$

It increases rapidly with time for  $m > 0$  and decays rapidly for  $m < 0$ . Notice that for a vacuum manifold, the material density could decrease with time if the gravitational constant is assumed to increase exponentially with time as  $G \propto e^{2mT}$ . The present accelerated expansion of the universe may be attributed to this ever-growing gravity. In fact, the exponential growth of the gravitational constant is one of the characteristics of inflation theory [17, 18]. It is important to note that a new type of vacuum is at work in our fractional scenario.

## 6. CONCLUSIONS

Despite the important question that arises “*what does the frictional force mean, in the vacuum?*” we showed that the new decaying lambda term leads to various modifications to the standard prediction of general relativity. There is nothing in the current notion of physical space that entails the presence of the decaying friction term. So long as the homogeneous space is represented by a differential manifold, and mass-energy is represented by fields on the manifold, it will be possible and even realistic to imagine an empty region of space. It may well be true that there is no physical procedure which can make a region of space totally empty, but this does not mean that it is impossible for space to be empty. If the material density decreases with time, then the universe tends to be an empty manifold and consequently, from equations (41) and (42), one can easily prove that the scale factor will evolve as  $a(T) \propto T^2$  and this is quite interesting. The empty universe accelerated with time, *i.e.* a decaying material universe creates an empty accelerating universe or an empty eternal accelerating universe is created from physical decaying friction. In Riemann geometry on manifolds and the geometry of the 4-dimensional FRW Lorentzian manifold, other applications and fractional features are under progress.

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7. Fractionally Differentiated Features is Chapter 5 about Fractionally Differentiated Features. 8. Data Labelling is Chapter 3 about The Triple-barrier Method. And 9. Meta-labeling is Chapter 3.6 on page 50. Approximate asymptotic distribution functions for unit-root and cointegration tests. `Journal of Business and Economic Statistics` 12, 167-176. Introduction to Advances in Financial Machine Learning by Lopez de Prado. Modelling of filtration in fractured porous media. A.V. Grigoriev in collaboration with P.N. Vabishchevich. Center of computational technologies NEFU. July 8, 2013. Center of computational technologies NEFU. c A.V. Grigoriev. July 8, 2013. Double porosity model. Introduction. Most oil reservoirs have fractured porous structure. Basic filtration flows are in cracks, and the basic fluid reserves are in porous blocks. These structural features of the fractured-porous environment are reflected in the corresponding model of filtration, often referred to as a double porosity model with the geometry of spacetime as this is specified by the metric of the Riemannian structure. Concerning the matter this is described in terms of various dynamical fields which are related to the matter via Einstein. The types of matter it can be described in the given geometry-model. For example if one considers the symmetries of the Friedmann Robertson Walker (FRW) model, which we shall discuss in the following, then the collineations. In conclusion, by symmetries in Cosmology we mean the work of Lie and Noether applied to the solution of the field equations of a given cosmological model - scenario. 2 Point transformations. On a manifold  $M$  with coordinates  $(t, q^a)$  one defines the jet space  $J^m(M)$  of order  $m$  over  $M$  to be a manifold.