International Mathematical Forum, Vol. 6, 2011, no. 67, 3315 - 3321

Analytical Solution to Intra-Phase Mass Transfer in Falling Film Contactors via Homotopy Perturbation Method

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Abstract

In this paper, mass transfer between gas and liquid phases in falling film units is investigated by Homotopy Perturbation Method (HPM). In a rather initial step of the analysis, Fourier expansion is employed to propose an apt first approximation. With less computational effort, HPM converges gracefully to a closed-form solution. Making reference to our prior works [*Australian Journal of Basic and Applied Sciences, 5(6): 1109-1115, 2011 and Australian Journal of Basic and Applied Sciences, 5(3): 337-345, 2011*], it is shown that the yielded solution corresponds to the exact analytical solution possibly attainable by other well-known methods like Adomian Decomposition, Differential Transform, and Separation of Variables.

Mathematics Subject Classification: 35A25; 80A20; 74K35

Keywords: Homotopy Perturbation Method, Falling liquid films, Molecular diffusion

1 Introduction

Differential equations seem to be omnipresent in our lives. They are frequently encountered in various fields of science and technology, for example transport phenomena, wave theory, hydrodynamics, aerodynamics, elasticity, quantum mechanics, thermal engineering, chemical engineering practices, etc. Mathematical literature abounds with numerous approaches to tackle such equations both analytically and numerically. Among them, Homotopy Perturbation Method (HPM) has gained reputation as being a powerful tool for solving functional equations and nonlinear differential equations in particular. Having been initially introduced by He in 1999 [1], HPM is a beneficial combination of perturbation and homotopy methods which robustly provides approximate analytical solutions. This method has been the subject of intense investigation/interest during recent years and many researchers have employed it in their works involving differential equations [2-6].

Falling liquid film phenomenon is of central significance in some chemical engineering unit operations including evaporation (e.g. in desalinization technology and juice concentration), gas absorption (e.g. ammonia scrubbing) and gas stripping (desorption of gases) [7]. Due to their favorable heat transfer properties, falling films have also found applications in refrigeration systems of nuclear fuel clusters [8]. Falling film reactors basically consist of a falling liquid film in which a gas-liquid chemical reaction takes place and are famous for their use in detergent industry (particularly the sulfonation process) [9].

It is the purpose of this study to apply HPM for treating a mathematical model representing mass transfer inside a falling liquid film apparatus. As it will be discussed, Fourier series expansion is exploited in course of the analysis to guide HPM towards a convergent closed form solution. It will be pointed that the resulted solution corresponds identically to what obtained from other analytical techniques. Of advantages of the followed approach, we can point to the computational simplicity and the fact that no linearization or discretization is required.

2 Model Assumptions and Formulation

To develop a model simulating the diffusion of a gas species A into a falling liquid film B, we make some engineering and simplifying assumptions as listed below:

1. The system is at steady-state condition (i.e. no property changes with time).
2. Fluid flows only downward along with the z-axis direction.
3. Physiochemical properties of the system are assumed to be constant.
4. Diffusion of the gas in the z-direction is negligible in comparison with its downward movement due to bulk flow.
5. The solid wall is non-diffusible and the motion of the liquid film is wave-free.
6. A constant average velocity is considered for the liquid flow (u).
7. No chemical reaction occurs in gas or liquid phases.

Setting up a mass balance over a rectangular element of infinitesimal dimensions within the liquid phase, while taking the above assumptions into account, gives a governing equation as:

$$\frac{D}{u}\frac{\partial^2 c}{\partial x^2} = \frac{\partial c}{\partial z} \tag{2.1}$$

subject to the following boundary conditions:

$$\begin{cases} c(0,z) = 0 \\ \frac{\partial c}{\partial x}(\delta, z) = 0 \\ c(x,0) = c_\dagger - c_* \end{cases} \tag{2.2}$$

where D, u, δ, $c_\dagger$, and c_* are molecular diffusion coefficient, average falling velocity, film thickness, concentration of component A within the liquid along the top end of the film, and concentration of component A on the gas-liquid interface, respectively. Also, obviously, c is the concentration of A measured above c_* value.

3 HPM in Brief

In this section, we present a short summary of the basics to HPM for the ease of the reader.

Let us consider the following general nonlinear functional equation over the domain Ω:

$$A(u) - f(r) = 0;\ r \in \Omega \tag{3.1}$$

which is subject to the subsequent boundary conditions:

$$B\left(u, \frac{\partial u}{\partial n}\right) = 0;\ r \in \Gamma \tag{3.2}$$

where A, B, $f(r)$ and Γ denote a general differential operator, a boundary operator, a known given function, and a boundary of the domain Ω, respectively. Without loss of generality, the operator A can be expressed as a summation of two new operators (a linear plus a nonlinear one). Therefore, the eq.(3.1) can possibly be rewritten as:

$$L(u) + N(u) - f(r) = 0 \tag{3.3}$$

According to the He's homotopy technique, a homotopy $v(r,p) : \Omega \times [0,1] \mapsto R$ can be constructed such that:

$$H(v,p) = (1-p)[L(v) - L(u_0)] + p[A(v) - f(r)] = 0 \tag{3.4}$$

or readily:

$$H(v,p) = L(v) - L(u_0) + p[L(u_0) + N(v) - f(r)] = 0 \tag{3.5}$$

where $p \in [0,1]$ is an embedding parameter and u_0 is the very first approximation of the solution to eq.(2.1) that has to satisfies the boundary conditions. From eq.(3.5), it apparently follows that:

$$H(v,0) = 0 \rightarrow L(v) - L(u_0) = 0 \tag{3.6}$$

and

$$H(v,1) = 0 \rightarrow L(v) + N(v) - f(r) = 0 \tag{3.7}$$

In other words, as the embedding parameter p monotonically goes from zero towards unity, v changes from the trivial solution to eq.(3.6), u_0, to the sought-after solution to eq.(3.1), u. It is natural to assume an expansive form for v like:

$$v = v_0 + pv_1 + p^2v_2 + p^3v_3 + ... \tag{3.8}$$

As noted, by letting the embedding parameter equal to unity, we achieve the solution:

$$u = \lim_{p \rightarrow 1} v = v_0 + v_1 + v_2 + v_3 + ... \tag{3.9}$$

The convergence of the latter series has been guaranteed [1].

4 Analysis by HPM

Following the principles of HPM, we construct the following homotopy for eq.(2.1) as:

$$(1-p)\left(\frac{\partial v}{\partial z} - \frac{\partial c_0}{\partial z}\right) + p\left(\frac{\partial v}{\partial z} - \frac{D}{u}\frac{\partial^2 u}{\partial x^2}\right) = 0 \tag{4.1}$$

or equally:

$$\frac{\partial v}{\partial z} - \frac{\partial c_0}{\partial z} = p\left(\frac{D}{u}\frac{\partial^2 v}{\partial x^2} - \frac{\partial c_0}{\partial z}\right) \tag{4.2}$$

where c_0 is the initial approximation of the solution and as mentioned above, it must satisfy all the boundary conditions. Accordingly, we propose the following candidate for c_0 as:

$$c_0 = \frac{4}{\pi}(c_\dagger - c_*) \sum_{n=0}^{\infty} \left\{ \frac{1}{2n+1} \sin\left(\frac{2n+1}{2\delta}\pi x\right) \right\} \tag{4.3}$$

In this way:

$$c_0(0, z) = 0 \tag{4.4}$$

$$\frac{\partial c_0}{\partial x}(\delta, z) = 0 \tag{4.5}$$

and according to Fourier sine expansion:

$$c_0(x, 0) = \frac{4}{\pi}(c_\dagger - c_*) \sum_{n=0}^{\infty} \left\{ \frac{1}{2n+1} \sin\left(\frac{2n+1}{2\delta} \pi x \right) \right\};\ 0 < x < \delta \tag{4.6}$$

By substituting $v = v_0 + pv_1 + p^2 v_2 + p^3 v_3 + ...$ into eq.(4.2) and collecting the terms with identical powers of p, it is yielded that:

$$p^0 \ : \ \frac{\partial v_0}{\partial z} - \frac{\partial c_0}{\partial z} = 0 \tag{4.7}$$

$$p^1 \ : \ \frac{\partial v_1}{\partial z} = \frac{D}{u} \frac{\partial^2 v_0}{\partial x^2} - \frac{\partial c_0}{\partial z};\ v_1(x, 0) = 0 \tag{4.8}$$

$$p^2 \ : \ \frac{\partial v_2}{\partial z} = \frac{D}{u} \frac{\partial^2 v_1}{\partial x^2};\ v_2(x, 0) = 0 \tag{4.9}$$

$$p^3 \ : \ \frac{\partial v_3}{\partial z} = \frac{D}{u} \frac{\partial^2 v_2}{\partial x^2};\ v_3(x, 0) = 0 \tag{4.10}$$

$$\vdots$$

$$p^i \ : \ \frac{\partial v_i}{\partial z} = \frac{D}{u} \frac{\partial^2 v_{i-1}}{\partial x^2};\ v_i(x, 0) = 0 \tag{4.11}$$

Thus:

$$v_0 = \frac{4}{\pi}(c_\dagger - c_*) \sum_{n=0}^{\infty} \left\{ \frac{1}{2n+1} \sin\left(\frac{2n+1}{2\delta} \pi x \right) \right\} \tag{4.12}$$

$$v_1 = -\frac{4}{\pi}(c_\dagger - c_*) \frac{D}{u} \sum_{n=0}^{\infty} \left\{ \left(\frac{1}{2n+1} \right)^2 \sin\left(\frac{2n+1}{2\delta} \pi x \right) \right\} z \tag{4.13}$$

$$v_2 = \frac{4}{\pi}(c_\dagger - c_*) \frac{D^2}{u^2} \sum_{n=0}^{\infty} \left\{ \left(\frac{1}{2n+1} \right)^4 \sin\left(\frac{2n+1}{2\delta} \pi x \right) \right\} \frac{z^2}{2} \tag{4.14}$$

$$v_3 = -\frac{4}{\pi}(c_\dagger - c_*)\frac{D^3}{u^3}\sum_{n=0}^{\infty}\left\{\left(\frac{1}{2n+1}\right)^6 \sin\left(\frac{2n+1}{2\delta}\,\pi x\right)\right\}\frac{z^3}{6} \tag{4.15}$$

$$\vdots$$

$$v_i = (-1)^i\frac{4}{\pi}(c_\dagger - c_*)\frac{D^i}{u^i}\sum_{n=0}^{\infty}\left\{\left(\frac{1}{2n+1}\right)^{2i} \sin\left(\frac{2n+1}{2\delta}\,\pi x\right)\right\}\frac{z^i}{i!} \tag{4.16}$$

By virtue of eq.(3.9), we have:

$$c = \lim_{p\to 1} v = \sum_{i=0}^{\infty} v_i = \frac{4}{\pi}(c_\dagger - c_*)\sum_{n=0}^{\infty}\left\{\frac{1}{2n+1}\sum_{i=0}^{\infty}\left\{\left(\frac{2n+1}{2\delta}\,\pi\right)^{2i}(-1)^i\,\frac{D^i}{u^i}\frac{z^i}{i!}\right\}\right. \\ \left.\sin\left(\frac{2n+1}{2\delta}\,\pi x\right)\right\} \tag{4.17}$$

which on rearrangement reads:

$$c = \frac{4}{\pi}(c_\dagger - c_*)\sum_{n=0}^{\infty}\left\{\frac{1}{2n+1}\sum_{i=0}^{\infty}\left\{\frac{\left(-\frac{D}{u}\left(\frac{2n+1}{2\delta}\,\pi\right)^2 z\right)^i}{i!}\right\}\sin\left(\frac{2n+1}{2\delta}\,\pi x\right)\right\} \tag{4.18}$$

Recognizing the Taylor series expansion of the exponential function in eq.(4.18), we finally obtain:

$$c = \frac{4}{\pi}(c_\dagger - c_*)\sum_{n=0}^{\infty}\left\{\frac{1}{2n+1}\sin\left(\frac{2n+1}{2\delta}\,\pi x\right)\exp\left(-\frac{D}{u}\left(\frac{2n+1}{2\delta}\,\pi\right)^2 z\right)\right\} \tag{4.19}$$

Upon scrutiny it is revealed that the preceding equation is the exact analytical solution to eq.(2.1) under the specified boundary conditions; see [10].

5 Conclusion

As the main part of this paper, a mathematical model describing the intraphase mass transfer phenomenon in falling liquid film contactors is treated by HPM. With the help of Fourier sine expansion, an apt initial approximation was selected to let HPM converge to a closed form solution of the model. In light of our previous works [10, 11], it is concluded that HPM has provided the exact solution to the problem absolutely identical to what obtained from Separation of Variables Method, Adomian Decomposition Method and Differential Transform Method.

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Received: July, 2011

We find the analytical solution of the proposed model by Homotopy perturbation method which is one of the best method for finding the solution of the nonlinear problem. By using this techniques, first, we solve the problem analytically and then compare the numerical results with other standards methods. We also justfy the numerical simulation and their results. Mostly nonlinear problem have upon some difficulties, and their solution is some time difficult to obtain. However, this techniques help us to obtain their approximate as well as analytical solution just after few perturbation terms. The key idea of perturbation methods is to interpret the solution to the model as a function of the state vector xt and of the parameter Ïƒ scaling the amount of uncertainty in the economy, that is, yt = g(xt, Ïƒ). (1.4).Â Given this interpretation, a perturbation methods nds a local approx-imation of the functions g and h. By a local approximation, we mean an approximation that is valid in the neighborhood of a particular point (xÂ¯, ÏƒÂ¯). Taking a Taylor series approximation of the functions g and h around the point (x, Ïƒ) = (xÂ¯, ÏƒÂ¯) we have (for the moment to keep the notation simple, letâ€™s assume that nx=ny=1).Â Symbolic Math can handle analytical derivatives. We wrote programs, that compute the analytical derivatives of f and evaluate them at the steady state. Fatoorehchi H, Abolghasemi H (2011) Analytical solution to intra-phase mass transfer in falling film contactors via homotopy perturbation method. Int Math Forum 6:3315â€“3321. Google Scholar. 23. Fatoorehchi H, Abolghasemi H (2011) Differential transform method to investigate mass transfer phenomenon to a falling liquid film system. Aust J Basic Appl Sci 5:337â€“345. Google Scholar. 24. Fatoorehchi H, Abolghasemi H (2011) Adomian decomposition method to study mass transfer within wetted wall columns. Aust J Basic Appl Sci 5:1109â€“1115. Google Scholar.