An ideal lattice is a pair \((I, b)\), where \(I\) is an ideal of a number field, and \(b\) is a lattice, satisfying an invariance relation (see §1 for the precise definition). Ideal lattices naturally occur in many parts of number theory, but also in other areas. They have been studied in special cases, but, as yet, not much in general. In the special case of integral ideal lattices, the survey paper Bayer-Fluckiger (1999) collects and slightly extends the known results.

The first part of the paper (see §2) concerns integral ideal lattices, and states some classification problems. In §3, a more general notion of ideal lattices is introduced, as well as some examples in which this notion occurs.

The aim of §4 is to define twisted embeddings, generalising the canonical embedding of a number field. This section, as well as the subsequent one, is devoted to positive definite ideal lattices with respect to the canonical involution of the real étale algebra generated by the number field. These are also called Arakelov divisors of the number field. The aim of §5 is to study invariants of ideals and also of the number field derived from Hermite type invariants of the sphere packings associated to ideal lattices. This again gives rise to several open questions.

1 Definitions, notation and basic facts

A lattice is a pair \((L, b)\), where \(L\) is a free \(\mathbb{Z}\)-module of finite rank, and \(b : L \times L \to \mathbb{R}\) is a non-degenerate symmetric bilinear form. We say that \((L, b)\) is an integral lattice if \(b(x, y) \in \mathbb{Z}\) for all \(x, y \in L\). An integral lattice \((L, b)\) is said to be even if \(b(x, x) \equiv 0 \pmod{2}\) for all \(x \in L\).

Let \(K\) be an algebraic number field. Let us denote by \(\mathcal{O}\) its ring of integers, and by \(D_K\) its discriminant. Let \(n\) be the degree of \(K\).
Set \( K_R = K \otimes Q R \). Then \( K_R \) is an étale \( R \)-algebra (i.e. a finite product of copies of \( R \) and \( C \)). Let us denote by \( N = N_{K_R/R} \) the norm, and by \( \text{Tr} = \text{Tr}_{K_R/R} \) the trace, of this étale algebra. Let \( \cdot : K_R \to K_R \) be an \( R \)-linear involution.

**Definition 1** An ideal lattice is a lattice \((I, b)\), where \( I \) is a (fractional) \( O \)-ideal and \( b : I \times I \to R \) is such that
\[
b(\lambda x, y) = b(x, \lambda y)
\]
for all \( x, y \in I \) and for all \( \lambda \in O \).

**Proposition 1** Let \( I \) be an \( O \)-ideal and let \( b : I \times I \to R \) be a lattice. Then the following are equivalent:

(i) \((I, b)\) is an ideal lattice;

(ii) there exists an invertible element \( \alpha \in K_R \) with \( \overline{\alpha} = \alpha \) such that
\[
b(x, y) = \text{Tr}(\alpha xy).
\]

**Proof** This follows from the fact that \( \text{Tr} : K_R \times K_R \to R \) is non-degenerate.

The rank of an ideal lattice is the degree \( n \) of the number field \( K \). As we shall see, the other basic invariants – determinant, signature – are also easy to determine. Let \( b : I \times I \to R \), \( b(x, y) = \text{Tr}(\alpha xy) \), be an ideal lattice.

**Proposition 2** We have
\[
|\det(b)| = N(I)^2N(\alpha)D_K.
\]

**Proof** Straightforward computation.

In order to determine the signature of ideal lattices, we need the notion of canonical involution (or complex conjugation) of an étale \( R \)-algebra.

**Definition 2** Let \( E \) be an étale \( R \)-algebra. We have \( E = E_1 \times \cdots \times E_r \times F_1 \cdots \times F_s \), where \( E_i \simeq R \) and \( F_i \simeq C \). We say that \( x = (x_1, \ldots, x_r) \) is positive, denoted by \( x > 0 \), if \( x_i \in R \) and \( x_i > 0 \) for all \( i = 1, \ldots, r \).

**Proposition 3** Let \( E \) be an étale \( R \)-algebra, and let \( * : E \to E \) be an involution. The following properties are equivalent:

(i) \( xx^* > 0 \) for all non-zero \( x \in E \);
(ii) the restriction of $\ast$ to $E_i$ is the identity for all $i = 1, \ldots, r$, and it is complex conjugation on $F_j$ for all $j = 1, \ldots, s$.

**Proof** This is immediate. □

In particular, this implies that for any étale $\mathbf{R}$-algebra $E$ there is exactly one involution such that $xx^\ast$ is positive for all non-zero $x \in E$.

**Definition 3** Let $E$ be an étale $\mathbf{R}$-algebra, and let $\ast : E \to E$ be an involution. We say that $\ast$ is the **canonical involution** of $E$ if and only if $xx^\ast > 0$ for all non-zero $x \in E$.

Let $C \subset \mathbf{K}_R$ be the maximal étale $\mathbf{R}$-subalgebra such that the restriction of $\ast$ to $C$ is the canonical involution of $C$. Set $c = \text{rank}(C)$.

We are now ready to determine the signature of an ideal lattice $(I, b)$:

$$b(x, y) = \text{Tr}(ax y)$$

Let $A \subset C$ be the maximal étale $\mathbf{R}$-subalgebra such that all the components of $a$ in $A$ are negative. Let $a = \text{rank}(A)$.

**Proposition 4** The signature of $(I, b)$ is $c - 2a$.

**Proof** This follows from the definitions. □

**Corollary 1** We have

$$\det(b) = (-1)^{n-1-a} N(I)^2 N(\alpha) D_K.$$

**Proof** This follows from Propositions 2 and 4. □

We now define some equivalence relations on the set of ideal lattices.

**Definition 4** Let $(I, b)$ and $(I', b')$ be two ideal lattices.

(i) We say that $(I, b)$ and $(I', b')$ are **isomorphic**, denoted by $(I, b) \simeq (I', b')$, if there exists $a \in \mathbf{K}^*$ such that $I' = aI$ and that $b'(ax, ay) = b(x, y)$ for all $x, y \in I$.

(ii) We say that $(I, b)$ and $(I', b')$ are **equivalent**, denoted by $(I, b) \equiv (I', b')$ (or simply $b \equiv b'$), if there exists an isomorphism of $\mathbf{Z}$-modules $f : I \to I'$ such that $b'(f(x), f(y)) = b(x, y)$ for all $x, y \in I$.

Recall that two ideals $I$ and $I'$ are said to be equivalent, denoted by $I \equiv I'$, if there exists $a \in \mathbf{K}^*$ such that $I' = aI$. 

Ideal Lattices

Proposition 5 Let \((I, b)\) and \((I', b')\) be two ideal lattices. Suppose that \((I, b) \simeq (I', b')\). Then \(I \equiv I'\) and \(b \equiv b'\).

Proof This is clear from the definitions.

2 Integral ideal lattices

We keep the notation of §1. In particular, \(K\) is an algebraic number field, and \(\mathcal{O}\) the ring of integers of \(K\). Let \(D_K\) be the different of \(K\), and let \(\Delta_K\) be its discriminant.

In this section we suppose that the chosen involution preserves \(K\). Let \(F\) be the fixed field of this involution. Then either \(K = F\) or \(K\) is a quadratic extension of \(F\).

The aim of this section is to study integral ideal lattices. Recall that a lattice \((L, b)\) is integral if \(b(x, y) \in \mathbb{Z}\) for all \(x, y \in L\); it is said to be even if \(b(x, x) \in 2\mathbb{Z}\) for all \(x \in L\).

Proposition 6 Let \(b : I \times I \to \mathbb{R}, b(x, y) = \text{Tr}(\alpha x \bar{y}),\) be an ideal lattice. Then \((I, b)\) is integral if and only if

\[ \alpha I \subset D_K^{-1}. \]

Proof This follows immediately from the definition.

For any non-zero integer \(d\), let us denote by \(\mathcal{L}_d\) the set of integral ideal lattices of determinant \(d\). Set \(C_d(K, \cdot) = \mathcal{L}_d/ \simeq\), and \(C_d(\cdot) = \mathcal{L}_d/ \equiv\). As usual, we denote by \(C(K)\) the ideal class group of \(K\).

We have two projection maps

\[ p_1 : C_d(K, \cdot) \to C(K), \]
\[ p_2 : C_d(K, \cdot) \to C_d(\cdot). \]

Several natural questions concerning ideal lattices can be formulated in terms of the sets \(C_d(K, \cdot), C_d(\cdot),\) and of the the projection maps \(p_1\) and \(p_2\). In particular, it is interesting to determine the images and the fibres of these maps. As we will see below, the results are far from complete, especially concerning the map \(p_2\).

Note that if an ideal lattice \((I, b)\) given by \(b(x, y) = \text{Tr}(\alpha x \bar{y})\) belongs to \(\mathcal{L}_d\), then \(N(\alpha I \bar{D}_K) = |d|\). Rather than fixing \(|D|\), it turns out that it is more convenient to fix the \(\mathcal{O}\)-ideal \(\alpha I \bar{I}\). The \(\mathcal{O}\)-ideal \(\alpha I \bar{I}\) will be called the norm of the ideal lattice \((I, b)\).
For any ideal $\mathcal{A}$, let $\mathcal{L}_A$ be the set of ideal lattices of norm $\mathcal{A}$. Set $\mathcal{C}_A(K, \cdot) = \mathcal{L}_A/\simeq$, and $\mathcal{C}_A(\cdot) = \mathcal{L}_A/\equiv$. Again, we have the projection maps
\[ p_1 : \mathcal{C}_A(K, \cdot) \to \mathcal{C}(K), \]
\[ p_2 : \mathcal{C}_A(K, \cdot) \to \mathcal{C}(\cdot). \]
Let us denote by $\mathcal{C}_A(K)$ the image of $p_1$.

The case where $\mathcal{A}$ is the ring of integers $\mathcal{O}$ of $K$ is especially interesting.

If $(I, b)$ and $(I', b')$ are two ideal lattices given by $b(x, y) = \text{Tr}(\alpha x \overline{y})$ and $b'(x, y) = \text{Tr}(\alpha' x \overline{y})$, then we define their product $(I, b)(I', b') = (II', bb')$ by setting $II'$ to be the product of the ideals $I$ and $I'$, and $(bb')(x, y) = \text{Tr}(\alpha \alpha' x \overline{y})$. If $(I, b)$ and $(I', b')$ have norm $\mathcal{O}$, then their product is again an ideal lattice of norm $\mathcal{O}$, hence we obtain a product on $\mathcal{C}_O(K, \cdot)$.

**Proposition 7** $\mathcal{C}_O(K, \cdot)$ is a group with respect to the above product.

*Proof* This is clear.

**Proposition 8** For any ideal $\mathcal{A}$, the set $\mathcal{C}_A(K, \cdot)$ is either empty, or a principal homogeneous space over the group $\mathcal{C}_O(K, \cdot)$.

*Proof* If $(I, b)$ has norm $\mathcal{A}$ and $(I', b')$ norm $\mathcal{O}$, then the product $(II', bb')$ has norm $\mathcal{A}$. Hence we obtain a structure of homogeneous space of $\mathcal{C}_A(K, \cdot)$ over $\mathcal{C}_O(K, \cdot)$. Let us check that it is a principal homogeneous space. This follows from the fact that if $\alpha I \overline{I} = \beta J \overline{J}$, then $\alpha \beta^{-1} (IJ^{-1})(IJ^{-1}) = O$.

We denote by $U_K$ be the group of units of $K$, by $U_F$ the group of units of $F$, and by $N_{K/F}$ the norm from $K$ to $F$.

**Proposition 9** (i) If the involution is trivial, then we have the exact sequence of groups
\[ 1 \to U_K / U_K^2 \to \mathcal{C}_O(K, \cdot) \xrightarrow{p_1} \mathcal{C}(K) \to 0. \]
(ii) If the involution is non-trivial, then we have the exact sequence of groups
\[ 1 \to U_F / N_{K/F}(U_K) \to \mathcal{C}_O(K, \cdot) \xrightarrow{p_1} \mathcal{C}(K) \to 0. \]

Some results about the order of $U_F / N_{K/F}(U_K)$ are given in Bayer (1982), §2.

Note that if $K = F$, then $\mathcal{C}_O(K)$ is the set of elements of order at most 2 in $\mathcal{C}(K)$. If $K \neq F$, then $\mathcal{C}_O(K)$ is the relative class group $\mathcal{C}(K/F)$, that is the kernel of the norm map $N_{K/F} : \mathcal{C}(K) \to \mathcal{C}(F)$. It is well-known that $N_{K/F}$ is
onto if \( K/F \) is ramified, and has cokernel of order 2 if \( K/F \) is unramified (see for instance Bayer 1982, proposition 1.2).

**Quadratic fields**

Suppose that \( K \) is a quadratic field, \( K = \mathbb{Q}(\sqrt{d}) \) where \( d \) is a square-free integer, and that the involution \( \bar{\cdot} : K \rightarrow K \) is given by \( \sqrt{d} = -\sqrt{d} \).

Let \( b : I \times I \rightarrow \mathbb{Z}, b(x, y) = \text{Tr}(\alpha x \bar{y}) \), be an integral, even ideal lattice. Then \( \alpha \in (1/N(I))/\mathbb{Z} \). In other words, \( b \) is an integral multiple of \( b_I : I \times I \rightarrow \mathbb{Z}, b_I(x, y) = \text{Tr}(\frac{1}{N(I)}x \bar{y}) \). Note that the quadratic form associated to \( b_I \) is \( q_I : I \rightarrow \mathbb{Z}, q_I(x) = N(x)/N(I) \). We have \( \det(b_I) = -DK \).

Set \( D = -DK \). Gauss defined a correspondence between ideal classes of \( K \) and binary quadratic forms of determinant \( D \), which sends an ideal \( I \) to the quadratic form \( q_I \). The precise statement will be given below, as well as a way of deriving it using the notion of ideal lattice.

Let us first note that \( C_O(K) = C_D(K) \). Indeed, if \( (I, b) \) is an ideal lattice of norm \( O \), then \( b = \pm b_I \), hence it has determinant \( D \). Conversely, an ideal lattice of determinant \( D \) has norm \( O \).

We can apply the results of the first part of this section to \( C_D(K) \). In particular, by Proposition 7, it is a group. Any ideal \( I \) satisfies \( (1/N(I))I\bar{I} = O \), hence \( C_O(K) = C(K) \). The involution is non-trivial, so we can apply Proposition 9(ii), and obtain the exact sequence

\[
1 \rightarrow \{\pm\}/N(U_K) \rightarrow C_D(K, \cdot) \xrightarrow{p_1} C(K) \rightarrow 0. \quad (\ast)
\]

We now need information concerning the set \( C_D(\cdot) \) and the map \( p_2 : C_d(K, \cdot) \rightarrow C_d(\cdot) \).

**Proposition 10** Let \( b \) be an even, binary lattice with determinant \( D \). Then

(i) There exists an ideal \( I \) in \( K = \mathbb{Q}(\sqrt{d}) \) such that \( (I, b) \) is an ideal lattice.
(ii) If \( I' \) is another ideal such that \( (I', b) \) is an ideal lattice, then \( I' \equiv I \) or \( I' \equiv \bar{I} \).
(iii) \( K \) is the only quadratic field over which \( b \) is an ideal lattice.

**Proof** Let \( (L, b) \) be an even binary lattice with determinant \( D \). Let \( R \) be the \( \mathbb{Z} \)-algebra associated to \( (L, b) \), that is,

\[
R = \{ (e, f) \in \text{End}(L) \times \text{End}(L) \mid b(ex, y) = b(x, fy) \}
\]

(cf. Bayer-Fluckiger 1987). Recall that the product of \( R \) is given by \( (e, f)(e', f') = (ee', ff') \), and that \( R \) is endowed with the involution \( (e, f) \mapsto \)
If \((L, b)\) is an ideal lattice over a quadratic field \(K' = \mathbb{Q}(\sqrt{\delta})\), then by definition \(b(\sqrt{\delta}x, y) = -b(x, \sqrt{\delta}y)\). Hence there exists \(e \in \text{End}(L)\) such that \((e, -e) \in R\).

Let us fix a \(\mathbb{Z}\)-basis of \(L\), and let
\[
\begin{pmatrix}
2A & B \\
B & 2C \\
\end{pmatrix}
\]
be the matrix of \(b\) in this basis. A straightforward computation shows that the matrix of \(e\) in this basis is an integral multiple of
\[
E = \begin{pmatrix}
B & 2C \\
-2A & -B \\
\end{pmatrix}.
\]
As the determinant of this matrix is \(D\), the field \(K'\) has discriminant \(-D\), hence \(K' = K\). This proves (iii).

Let \(O = \mathbb{Z}[w]\) be the ring of integers of \(K\), with \(w = \sqrt{d}\) if \(d \equiv 2, 3 \pmod{4}\), and \(w = (1 - \sqrt{d})/2\) if \(d \equiv 1 \pmod{4}\). Letting \(w\) act by multiplication with \(E\) if \(d \equiv 2, 3 \pmod{4}\), and by multiplication with \((1 - E)/2\) if \(d \equiv 1 \pmod{4}\) provides \(L\) with a structure of \(O\)-module. This proves that \(b\) is an ideal lattice over \(K\), hence assertion (i). Moreover, we see that the only other way of making \(w\) act on \(L\) is by replacing \(E\) with \(-E\). This proves (ii).

**Remark 6** Note that the proof of Proposition 10 is constructive: given an even, binary lattice \(b\) one constructs the ideals \(I\) and \(I'\) over which \(b\) is an ideal lattice.

The following are immediate consequences of Proposition 10:

**Corollary 2** The set \(C_D(\cdot)\) is equal to the set of similarity classes of even binary lattices with determinant \(D\).

**Corollary 3** If \((I, b)\) and \((I', b')\) are two ideal lattices over \(K\) such that \(b \simeq b'\), then either \(I' \equiv I\) or \(I' \equiv \overline{I}\).

In order to recover the usual statement of Gauss’ correspondence between classes of binary quadratic forms and lattices, we need slightly different equivalence relations.

**Definition 5** (i) We say that two ideal lattices \((I, b)\) and \((I', b')\) are strictly isomorphic if there exists \(a \in K^*\) with \(N(a) > 0\) such that \(I' = aI\) and \(b'(ax, ay) = b(x, y)\).
(ii) We say that two lattices \((L, b)\) and \((L', b')\) with \(L \otimes \mathbb{Z} \mathbb{Q} = L' \otimes \mathbb{Z} \mathbb{Q}\) are \textit{strictly equivalent} if there exists a \(\mathbb{Z}\)-linear isomorphism \(f : L \rightarrow L'\) with \(\det(f) > 0\) such that \(b'(fx, fy) = b(x, y)\).

(iii) We say that two ideals \(I\) and \(I'\) are \textit{strictly equivalent} if there exists \(a \in \mathbb{K}^*\) with \(N(a) > 0\) such that \(I' = aI\).

Let us denote by \(C_s D(\mathbb{K}, 0)\), respectively \(C_s D(0)\), the set of strict isomorphism classes, respectively the set of strict equivalence classes, of ideal lattices of determinant \(D\). Let us denote by \(C_s(\mathbb{K})\) the strict (or narrow) ideal class group.

Let us denote by \(L^+ D\) be the set of positive-definite ideal lattices of determinant \(D\), and set \(C^+_D(\mathbb{K}, \cdot) = L^/ \simeq C^+_D(\cdot) = L^/ \equiv\).

If \(\mathbb{K}\) is an imaginary quadratic field, then the exact sequence (*) yields the isomorphism
\[ C^+_D(\mathbb{K}, \cdot) \simeq C(\mathbb{K}). \]

On the other hand, if \(\mathbb{K}\) is a real quadratic field, then we obtain from (*) the isomorphism
\[ C^+_D(\mathbb{K}, \cdot) \simeq C^s(\mathbb{K}). \]

Note that if two ideal lattices are strictly isomorphic, then the corresponding ideals are strictly equivalent. This follows from the proof of Proposition 10.

Using this, we obtain the well-known fact that the ideal class group \(C(\mathbb{K})\) is isomorphic to the set of strict equivalence classes of positive-definite, even binary lattices of determinant \(D\) if \(\mathbb{K}\) is imaginary; the strict ideal class group \(C^s(\mathbb{K})\) is isomorphic to the set of strict equivalence classes of even binary lattices of determinant \(D\) if \(\mathbb{K}\) is real.

\textbf{Cyclotomic fields of prime power conductor}

Let \(p\) be a prime number, \(r \geq 1\) an integer, and let \(\zeta_{p^r}\) be a primitive \(p^r\)th root of unity. Suppose that \(\mathbb{K} = \mathbb{Q}(\zeta_{p^r})\), the corresponding cyclotomic field, and that the involution is complex conjugation. Recall that \(\mathcal{O} = \mathbb{Z}[\zeta_{p^r}]\), that there is exactly one ramified ideal \(P\) in the extension \(\mathbb{K}/\mathbb{Q}\), and that \(N(P) = p\). Hence the different \(\mathcal{D}_K\) is a power of \(P\). Let \(D = |\mathcal{D}_K|\).

\textbf{Proposition 11} We have \(C_\mathcal{O}(\mathbb{K}, \cdot) = C_D(\mathbb{K}, \cdot)\).

\textbf{Proof} Let \((I, b)\) be an integral ideal lattice with norm \(\alpha I \overline{I}\) and determinant \(D\). Then \(N(\alpha I \overline{I}) = 1\), and \(\alpha I \overline{I} \subset \mathcal{D}_K\). As \(\mathcal{D}_K\) is a power of the single prime ideal \(P\), this implies that \(\alpha I \overline{I} = \mathcal{O}\). This shows that \(C_D(\mathbb{K}, \cdot) \subset C_\mathcal{O}(\mathbb{K}, \cdot)\). Conversely, if \((I, b)\) is an ideal lattice with norm \(\mathcal{O}\), then the determinant of
$(I, b)$ is $\pm D$. By Corollary 1, the determinant is positive, hence $\det(b) = D$. This proves the other inclusion, hence the proposition is proved.

The fixed field of the involution is the maximal totally real subfield $F$ of $K$. Applying Proposition 9(ii) and the remarks following, we have the exact sequence

$$1 \to U_F / N_{K/F}(U_K) \to \mathcal{C}_D(K, \cdot) \xrightarrow{p_1} \mathcal{C}(K/F) \to 0.$$ 

The order of $U_F / N_{K/F}(U_K)$ is $2^n$, where $n = [K : \mathbb{Q}]$, cf. Bayer (1982), proposition 2.3, and example 2.5. The order of $\mathcal{C}(K/F)$, called the relative class number of $K$, is known in many cases, see for instance Washington (1982).

Let $\mathcal{L}_D^+$ be the set of positive-definite ideal lattices of determinant $D$, and let $\mathcal{C}_D^+(K, \cdot)$ be the set of isomorphism classes of these lattices, that is $\mathcal{C}_D^+(K, \cdot) = \mathcal{L}_D^+ / \simeq$.

Let us denote by $U_F^+$ the set of totally positive units of $F$. Then we have the exact sequence

$$1 \to U_F^+ / N_{K/F}(U_K) \to \mathcal{C}_D^+(K, \cdot) \xrightarrow{p_1} \mathcal{C}(K/F).$$

If moreover the relative class number of $K$ is odd, then $U_F^+ = N_{K/F}(U_K)$ (cf. Shimura 1977, proposition A2), and we have the isomorphism $\mathcal{C}_D^+(K, \cdot) \simeq \mathcal{C}(K/F)$.

**Examples over cyclotomic fields**

Let $m$ be an integer, $\zeta_m$ a primitive $m$th root of unity, and suppose that $K = \mathbb{Q}(\zeta_m)$ is the corresponding cyclotomic field. It is not known in general which are the ideal lattices over $K$, but many examples are available. For instance, the root lattices $A_{p-1}$ (where $p$ is a prime) are ideal lattices for $m = p$, the root lattice $E_8$ is an ideal lattice for $m = 9$ and $E_8$ for $m = 15, 20, 24$. A complete description of root lattices that are ideal lattices over cyclotomic fields is given in Bayer-Fluckiger & Martinet (1994), A2. Moreover, the Coxeter–Todd lattice is an ideal lattice for $m = 21$, and the Leech lattice for $m = 35, 39, 52, 56, 84$ (cf. Bayer-Fluckiger 1984, 1994 and the survey in Bayer-Fluckiger 1999). Let us also point out the computations in higher rank cases of Bachoc & Batut (1992), of Batut, Quebbemann & Scharlau (1995), as well as the construction by Nebe (1998) of a unimodular rank-48 lattice with minimum 6 that is an ideal lattice for $m = 65$. Finally, in Bayer-Fluckiger (2000) examples of modular ideal lattices are given when $m$ is not a power of a prime $p$ with $p \equiv 1 \pmod{4}$. 

3 Generalization and examples

The aim of this section is to indicate a possible generalization of the notion of integral ideal lattice, and the usefulness of this notion in some parts of algebra and topology.

Let $A$ be a finite dimensional $\mathbb{Q}$-algebra with a $\mathbb{Q}$-linear involution $\tau : A \to A$. Let $O$ be an order of $A$ which is invariant under the involution. In this context, an integral ideal lattice will be a pair $(I, b)$, where $I$ is a (left) $O$-ideal and $b : I \times I \to \mathbb{Z}$ is a lattice such that

$$b(\lambda x, y) = b(x, \lambda y)$$

for all $x, y \in I$ and for all $\lambda \in O$. This is clearly a generalization of the notion of §2, where $A = K$ was a number field and $O$ the ring of integers of $K$.

**Proposition 12** Suppose that $A$ is semi-simple. Let $(I, b)$ be an ideal lattice. Then there exists $\alpha \in A$ such that $b(x, y) = \text{Tr}(\alpha xy)$.

**Proof** This follows from the fact that, as $A$ is semi-simple, $\text{Tr} : A \times A \to \mathbb{Q}$ is non-degenerate.

Integral ideal lattices naturally appear in several parts of mathematics. In the two examples below $A$ is commutative. However, there are also very interesting examples where $A$ is non-commutative, for instance in the study of polarized abelian varieties.

**Knot theory**

This example concerns odd-dimensional knots and their algebraic invariants. See Kearton (2000) or Kervaire & Weber (1978) for surveys of the relevant definitions and properties.

Let $k$ be a positive integer, $k \equiv 3 \pmod{4}$. Let $\Sigma^k \subset S^{k+2}$ be a fibred knot, and let $\Delta \in \mathbb{Z}[X]$ be the Alexander polynomial of $\Sigma^k$. Then $\Delta$ is monic, we have $\Delta(X) = X^{\deg(\Delta)} \Delta(X^{-1})$ and $\Delta(1) = \pm 1$. Suppose moreover that $\Delta$ has no repeated factors.

Let $A = \mathbb{Q}[X]/(\Delta) = \mathbb{Q}(\tau)$. Then $A$ is a finite-dimensional, semi-simple $\mathbb{Q}$-algebra. Let $\tau : A \to A$ be the $\mathbb{Q}$-linear involution induced by $\tau = \tau^{-1}$. Set $O = \mathbb{Z}[X]/(\Delta) = \mathbb{Z}[\tau]$. Then $O$ is an order of $A$.

Let $M^{k+1}$ be a minimal Seifert surface of $\Sigma^k$. Set $r = k + 1/2$, and set

$$I = H_r(M^{k+1}, \mathbb{Z})/(\text{torsion}).$$
Then $I$ is a rank one $O$-module, hence isomorphic to an $O$-ideal. The intersection form $b : I \times I \to \mathbb{Z}$ is a symmetric bilinear form of determinant $\Delta(1) = \pm 1$. Moreover, $(I, b)$ is an ideal lattice. Indeed, the monodromy of the fibration induces an isomorphism $t : I \to I$ that preserves the intersection form. In other words, we have $b(tx, ty) = b(x, y)$ for all $x, y \in I$. The Alexander polynomial is also the characteristic polynomial of $t$, hence $t$ acts as $\tau$ on $I$. We get $b(tx, y) = b(x, \tau^{-1}y)$. Noting that $\tau^{-1} = \tau$, we see that $b(\lambda x, y) = b(x, \lambda y)$ for all $x, y \in I$ and all $\lambda \in O$. Hence $(I, b)$ is an ideal lattice.

We define the sets $C_{\Delta(1)}(O, \cdot)$, $C_{\Delta(1)}(\cdot \cdot)$ and $C_{\Delta(1)}(O)$, as well as the projection maps $p_1 : C_{\Delta(1)}(O, \cdot) \to C_{\Delta(1)}(O)$, $p_2 : C_{\Delta(1)}(O, \cdot \cdot) \to C_{\Delta(1)}(\cdot \cdot)$ as in §2.

These have topological significance. Indeed, the class of $(I, b)$ in $C_{\Delta(1)}(O, \cdot \cdot)$ is an invariant of the isotopy class of the knot. Its image by $p_1$ is the Alexander module of the knot, and its image by $p_2$ is an invariant of the homeomorphism class of the minimal Seifert surface.

Moreover, if we suppose that the knot is simple, then these invariants are complete. The usefulness of this approach to solve concrete problems in knot theory is illustrated by several examples in Kearton (2000) and Bayer-Fluckiger (1999).

**Symmetric, skew-symmetric and orthogonal matrices with a given characteristic polynomial**

Let $f \in \mathbb{Z}[X]$ be a monic polynomial, and set $A = \mathbb{Q}[X]/(f)$, $O = \mathbb{Z}[X]/(f)$. Then $A$ is a finite-dimensional $\mathbb{Q}$-algebra, and $O$ is an order of $A$. The involution $\cdot : A \to A$. will be the identity (trivial involution).

Let $b_0 : L \times L \to \mathbb{Z}$ be the unit lattice. In other words, there exists a basis of $L$ in which the matrix of $b_0$ is the identity matrix.

The following proposition is (essentially) due to Bender (1968):

**Proposition 13** There exists an integral symmetric matrix with characteristic polynomial $f$ if and only if $b_0$ is an ideal lattice.

**Proof** Let $M \in M_n(\mathbb{Z})$ such that $M^t = M$ and that the characteristic polynomial of $M$ is $f$. Let $L$ be a free $\mathbb{Z}$-module of rank $n$, and let $(e_1, \ldots, e_n)$ be a basis of $L$ in which $b_0$ is the identity matrix. Let $m : L \to L$ be the endomorphism given by the matrix $M$ in this matrix. Let us endow $L$ with the $O$-module structure induced by $m$ (that is, the action of $X$ is given by $m$). As
M is symmetric, we have \( b_0(mx, y) = b_0(x, my) \) for all \( x, y \in L \). This proves that \( b_0 \) is an ideal lattice.

Conversely, suppose that \( b_0 : I \times I \to \mathbb{Z} \) is an ideal lattice. Let us denote by \( m : I \to I \) the endomorphism given by the image of \( X \) in \( \mathcal{O} \). Then the characteristic polynomial of \( m \) is \( f \). As \( (I, b_0) \) is an ideal lattice, we have

\[
 b_0(mx, y) = b_0(x, my) \tag{**}
\]

for all \( x, y \in I \). Let \( (e_1, \ldots, e_n) \) be a basis with respect to which the matrix of \( b_0 \) is the identity matrix. The relation (**) then shows that \( M' = M \). This concludes the proof of the proposition.

Similar results can be proved for skew-symmetric and orthogonal matrices with given characteristic polynomial. In these cases, the involution is non-trivial. It is induced by \( X \mapsto -X \) in the first case, and by \( X \mapsto X^{-1} \) in the second.

### 4 Real ideal lattices

So far, we have considered lattices up to isomorphism, rather than embedded in an euclidean space. However, it is often important to find suitable embeddings, and this will be the subject matter of this section.

Let \( K \) be a number field of degree \( n \), and let \( \mathcal{O} \) be its ring of integers. Let \( \cdot : K_\mathbb{R} \to K_\mathbb{R} \) be the canonical involution (cf. Proposition 4). In this section and the next, all lattices will be supposed positive-definite.

Suppose that the number field \( K \) has \( r_1 \) real embeddings, and \( r_2 \) pairs of imaginary embeddings. We have \( n = r_1 + 2r_2 \). Let \( \sigma_1, \ldots, \sigma_{r_1} \) be the real embeddings, and let \( \sigma_{r_1+1}, \ldots, \sigma_{r_2} \) be non-conjugate imaginary embeddings.

Let \( \alpha = (\alpha_1, \ldots, \alpha_n) \) be a positive element of \( K_\mathbb{R} \), in other words \( \alpha_i \) is real and positive for all \( i \). Let \( \sigma_\alpha : K \to \mathbb{R}^n \) be the embedding defined by

\[
 \sigma_\alpha(x) = \left( \sqrt{\alpha_1} x_1, \ldots, \sqrt{\alpha_{r_1}} x_{r_1}, \sqrt{2\alpha_{r_1+1}} \Re(x_{r_1+1}), \sqrt{2\alpha_{r_1+1}} \Im(x_{r_1+1}), \ldots, \sqrt{2\alpha_{r_2}} \Re(x_{r_2}), \sqrt{2\alpha_{r_2}} \Im(x_{r_2}) \right),
\]

where \( x_i = \sigma_i(x) \), \( \Re \) denotes the real part and \( \Im \) the imaginary part. Note that this definition differs slightly from the one in Bayer-Fluckiger 1999, Definition 5.1).

**Proposition 14** For any ideal \( I \) of \( K \) and any positive \( \alpha \in K_\mathbb{R} \), the lattice \( \sigma_\alpha(I) \subset \mathbb{R}^n \) is an ideal lattice. Conversely, for any ideal lattice \( (I, b) \) there exists an \( \alpha \in K_\mathbb{R} \) such that the ideal lattice \( \sigma_\alpha(I) \) is isomorphic to \( (I, b) \).
It is clear that $\sigma_\alpha(I) \subset \mathbb{R}^n$ is a lattice. A straightforward computation shows that $\sigma_\alpha(I)$ is isomorphic to the lattice $b : I \times I \to \mathbb{R}$ given by $b(x, y) = \text{Tr}(\alpha x \overline{y})$. Hence it is an ideal lattice. Conversely, let $(I, b)$ be an ideal lattice given by $b(x, y) = \text{Tr}(\alpha x \overline{y})$, then $\sigma_\alpha(I)$ is isomorphic to $(I, b)$. 

The above proposition is useful in information theory. Indeed, let us recall that if $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, then the diversity of $x$, denoted by $\text{div}(x)$, is the number of non-zero $x_i$s. Let $L \subset \mathbb{R}^n$ be a lattice. One defines the diversity of $L$, denoted $\text{div}(L)$, by

$$\text{div}(L) = \min\{\text{div}(x) \mid x \in L, x \neq 0\}.$$ 

Lattices of high diversity tend to perform better than the Rayleigh fading channel (see Boutros et al. 1996, Boutros & Viterbo 1998). The following proposition is proved in Bayer-Fluckiger (1999) in some special cases:

**Proposition 15** Any ideal lattice can be embedded in an euclidean space with diversity $r_1 + r_2$.

**Proof** Let $I$ be an ideal and let $\alpha \in K\mathbb{R}$ be totally real and totally positive. It is easy to see that the lattice $\sigma_\alpha(I)$ has diversity $r_1 + r_2$. By Proposition 14, any ideal lattice can be realised under this form, so the proposition is proved. 

## 5 Arakelov invariants

We keep the notation of §4. In particular, $K$ is a number field of degree $n$, and $\mathcal{O}$ its ring of integers. The involution $- : K\mathbb{R} \to K\mathbb{R}$ is again the canonical involution, and all lattices in this section are supposed positive-definite.

A positive-definite ideal lattice with respect to the canonical involution is also called an Arakelov divisor of the number field $K$. Such a lattice defines a sphere packing in $\mathbb{R}^n$, and the density and thickness of this packing are natural invariants of the lattice. We call these here Arakelov invariants.

**Definition 6** Let $(L, b)$ be a lattice, and set $q(x) = b(x, x)$. Let $V = L \otimes_{\mathbb{Z}} \mathbb{R}$.

(i) The minimum of $b$ is defined by $\min(b) = \inf\{q(x) \mid x \in L, x \neq 0\}$.

(ii) The maximum of $b$ is by definition

$$\max(b) = \sup\{\lambda \in \mathbb{R} \mid \forall x \in V, \exists y \in L \text{ with } q(x - y) \leq \lambda\}.$$ 

Note that $R = \sqrt{\max(b)}$ is the covering radius of $b$, and $r = \sqrt{\min(b)}/2$ is
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its packing radius. The thickness and the density of the sphere packing associated to \( b \) are defined in terms of these quantities (see Conner & Perlis 1984, chapters I and II). We will here use the related notions of Hermite invariants.

**Definition 7** Let \((L, b)\) be a lattice. The Hermite invariants are defined as follows:

1. \( \gamma(b) = \frac{\text{min}(b)}{\det(b)^{1/n}} \)
2. \( \tau(b) = \frac{\text{max}(b)}{\det(b)^{1/n}} \)

Note that the invariant \( \gamma(b) \) is classical, but \( \tau(b) \) is not. However, it is a natural invariant, related to the thickness as the \( \gamma \)-invariant is to the density. It is also a useful invariant, as we will see below.

It is also useful to consider the best Hermite invariants for lattices of a given rank.

**Definition 8**

1. \( \gamma_n = \text{sup}\{ \gamma(b) | \text{rank}(b) = n \} \)
2. \( \tau_n = \text{inf}\{ \tau(b) | \text{rank}(b) = n \} \)

These notions provide us with invariants of the ideal classes of \( K \), and of \( K \) itself.

**Definition 9** Let \( I \) be an ideal. Set

1. \( \gamma_{\text{min}}(I) = \text{inf}\{ \gamma(b) | (I, b) \text{ is an ideal lattice } \} \)
2. \( \gamma_{\text{max}}(I) = \text{sup}\{ \gamma(b) | (I, b) \text{ is an ideal lattice } \} \)
3. \( \tau_{\text{min}}(I) = \text{inf}\{ \tau(b) | (I, b) \text{ is an ideal lattice } \} \)
4. \( \tau_{\text{max}}(I) = \text{sup}\{ \tau(b) | (I, b) \text{ is an ideal lattice } \} \)

As equivalent ideals carry isomorphic ideal lattices, these are actually invariants of the ideal classes. It is natural to also use these notions to define invariants of the field \( K \), \( \gamma_{\text{min}}(K) \), \( \gamma_{\text{max}}(K) \), \( \tau_{\text{min}}(K) \), and \( \tau_{\text{max}}(K) \). Recall that \( D_K \) is the discriminant of \( K \). If \( I \) is an ideal, let us denote by \( \text{min}(I) \) the smallest norm of an integral ideal equivalent to \( I \).

**Proposition 16** Let \( I \) be an ideal. Then

\[ \gamma_{\text{min}}(I) \geq \frac{n}{|D_K|^{1/n}} \text{min}(I)^{2/n}. \]
Proof Let \((I, b)\) be an ideal lattice. Set \(q(x) = b(x, x)\). We have \(q(x) = \text{Tr}(\alpha x^2)\) for some positive \(\alpha \in K\). Recall that \(\det(b) = N(\alpha)N(I)^2|D_K|\).

By the inequality between the arithmetic and geometric means, we have
\[
\text{Tr}(\alpha x^2) \geq n \frac{N(\alpha x^2)^{1/n}}{|D_K|^{1/n}} \geq n \frac{N(x)^{2/n}}{|D_K|^{1/n}} - \frac{2}{n} N(x)^{2/n}.
\]
Hence
\[
\frac{q(x)}{\det(b)^{1/n}} \geq \frac{n}{|D_K|^{1/n}} \left(\frac{N(x)}{N(I)}\right)^{2/n}.
\]
and this implies that
\[
\frac{\min(b)}{\det(b)^{1/n}} \geq \frac{n}{|D_K|^{1/n}} \min(I)^{2/n}.
\]
As \(\gamma(b) = \min(b)/\det(b)^{1/n}\), the proposition is proved.

**Corollary 4** We have
\[
\gamma_{\min}(O) = \frac{n}{|D_K|^{1/n}}.
\]

**Proof** By Proposition 16 we have \(\gamma_{\min}(O) \geq n/|D_K|^{1/n}\). On the other hand, the ideal lattice \(b : O \times O \to \mathbb{Z}\) given by \(b(x, y) = \text{Tr}(x^2)\) has minimum \(n\) and determinant \(|D_K|\). Hence the equality holds.

Note that this also implies that \(\gamma_{\min}(O) = \gamma_{\min}(K)\).

**Corollary 5** For any ideal \(I\) we have
\[
\frac{\gamma_{\min}(I)}{\gamma_{\min}(O)} \geq \min(I)^{2/n}.
\]

**Proof** This follows from Definition 9 and Proposition 16.

The following is an immediate consequence of Corollary 5.

**Corollary 6** Let \(I\) be an ideal. If there exists an ideal lattice \((I, b)\) with \(\gamma(b) = \gamma_{\min}(O)\), then \(I\) is principal.

Recall that the field \(K\) is said to be Euclidean with respect to the norm if for every \(a, b \in O, b \neq 0\), there exist \(c, d \in O\) such that \(a = bc + d\) and \(|N(d)| < |N(b)|\).

**Proposition 17** Suppose that \(\tau_{\min}(O) < \gamma_{\min}(O)\). Then \(K\) is Euclidean with respect to the norm.
Proof The argument of Bayer-Fluckiger (1999), proposition 4.1 gives the desired result. 

Example 1 Let $K = \mathbb{Q}(\zeta_{15})$. We have $n = 8$, $D_K = 3^45^6$. Hence $\gamma_{\text{min}}(O) = 8/3^{1/2}5^{3/4}$. The root lattice $E_8$ is an ideal lattice over $O$ (see for instance Bayer-Fluckiger 1999). We have $\det(E_8) = 1$. The covering radius of $E_8$ is 1 (cf. Conway & Sloane 1988), hence $\max(E_8) = 1$. This implies that $\tau(E_8) = 1$, therefore $\tau_{\text{min}}(O) \leq 1$. This implies that $\tau_{\text{min}}(O) < \gamma_{\text{min}}(O)$, so by Proposition 17, $K$ is Euclidean with respect to the norm. We have $\gamma(E_8) = 2$. It is known that $\gamma(E_8) = \gamma_8$ (see Craif 1978), hence $\gamma_{\text{max}}(O) = 2$. Summarising, we have

$$\tau_{\text{min}}(O) \leq 1 < \gamma_{\text{min}}(O) < \gamma_{\text{max}}(O) = 2.$$ 

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