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Abstract. We show that there exists a new connection between identities satisfied by mock theta functions and special case of denominator identities for affine Lie superalgebras.

1. Introduction

In 1920, S. Ramanujan listed 17 mock theta functions of order 3, 5 and 7 together with identities satisfied by them in his last letter to G. H. Hardy. In the letter, he did not describe a formal definition of "order" nor prove these identities. However, for 3rd and 5th order, we can see that each identity consists of mock theta functions with the same order [2], [16]. Ramanujan's assertion about 7th order mock theta functions is that they are not related to each other. Some identities for order 6 , 8 and 10, which also consists of mock theta functions with the same order, were proved by G. E. Andrews, D. Hickerson, B. Gordon, R. J. McIntosh and Y.-S. Choi [3], [5], [6], [9]. In [13], mock theta functions 3rd order $\chi(q)$ and 6th order $\gamma(q)$ are related to each other. Recently, K. Bringmann and K. Ono use the theories of modular forms to understand mock theta functions[4].

In this paper, we give a new view of some typical identities satisfied by mock theta functions. It is shown that these identities are obtained by specializing the denominator identities for affine Lie superalgebras in the case $\widehat{A}(1,0)$ and $\widehat{B}(1,1)$.
We introduce the standard notation and some mock theta functions which will be used in this paper.

Definition 1. Let q be a complex number such that $|q| < 1$. We define the q-shifted factorial for all integers n by

$$(a)_\infty = (a;q)_\infty := \prod_{i=0}^{\infty}(1-aq^i) \qquad \text{and} \qquad (a)_n = (a;q)_n := \frac{(a;q)_\infty}{(aq^n;q)_\infty}.$$

For brevity, we employ the usual notation

$$(x_1,\ldots,x_r)_\infty = (x_1,\ldots,x_r;q)_\infty := (x_1)_\infty \cdots (x_r)_\infty.$$

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Moreover, we define

$$j(x,q) := (x, q/x, q)_\infty = (x)_\infty (q/x)_\infty (q)_\infty$$

for all $x \in \mathbf{C}^*$.

By Jacobi's triple product identity[1], [8], we have

$$j(x,q) = \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{n(n-1)}{2}} x^n.$$

Definition 2. (mock theta functions of order 3)

$$\begin{aligned}
f(q) &:= \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q)_n^2}, \\
\chi(q) &:= 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{\prod_{m=1}^{n}(1 - q^m + q^{2m})}, \\
\omega(q) &:= \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(q;q^2)_{n+1}^2}, \\
\rho(q) &:= \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{\prod_{m=0}^{n}(1 + q^{2m+1} + q^{4m+2})}.
\end{aligned}$$

The following mock theta functions of order 8 are found by B. Gordon and R. J. McIntosh[9].

Definition 3. (mock theta functions of order 8)

$$\begin{aligned}
U_0(q) &:= \sum_{n=0}^{\infty} \frac{q^{n^2}(-q;q^2)_n}{(-q^4;q^4)_n}, \\
U_1(q) &:= \sum_{n=0}^{\infty} \frac{q^{(n+1)^2}(-q;q^2)_n}{(-q^2;q^4)_{n+1}}, \\
V_1(q) &:= \sum_{n=0}^{\infty} \frac{q^{(n+1)^2}(-q;q^2)_n}{(q;q^2)_{n+1}} = \sum_{n=0}^{\infty} \frac{q^{2n^2+2n+1}(-q^4;q^4)_n}{(q;q^2)_{2n+2}}.
\end{aligned}$$

In Section 1, we shall prove a following new identity for 3rd order mock theta functions in Theorem 1.1 and then we shall show that the identity is equal to the special case of type $\widehat{B}(1,1)$.

Theorem 1.1. *For 3rd order mock theta functions $\rho(q)$ and $\omega(q)$, the following relation holds.*

$$(q^2;q^2)_\infty \left((\rho(q)+\rho(-q)) + \frac{1}{2}(\omega(q)+\omega(-q)) \right) = \frac{3\ (q^{12};q^{12})_\infty(-q^{12};q^{24})_\infty(-q^{12};q^{24})_\infty(q^{24};q^{24})_\infty}{(q^6;q^{12})_\infty}.$$

In Section 2, we shall see that the identity

$$(q)_\infty (4\chi(q) - f(q)) = 3\frac{(q^3;q^3)^2_\infty}{(-q^3;q^3)^2_\infty} \tag{1.1}$$

is equal to the case of type $\widehat{A}(1,0)$. This identity (1.1) is found in Ramanujan's last letter. We also see that some identities for mock theta functions are derived from a specialization of the denominator identity.

We give generalized Lambert series for mock theta functions in order to prove these identities satisfied by mock theta functions. By expressing mock theta functions with the generalized Lambert series, we can handle the mock theta functions more easily. The following generalized Lambert series for 3rd order mock theta functions are obtained by G. N. Watson[16].

$$(q)_\infty\ f(q) = 1 + 4\sum_{n=1}^{\infty} \frac{(-1)^n q^{n(3n+1)/2}}{1+q^n}, \tag{1.2}$$

$$(q)_\infty\ \chi(q) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n (1+q^n) q^{n(3n+1)/2}}{1-q^n+q^{2n}}, \tag{1.3}$$

$$(q^2;q^2)_\infty\ \omega(q) = \sum_{n=0}^{\infty} \frac{(-1)^n (1+q^{2n+1}) q^{3n(n+1)}}{1-q^{2n+1}}, \tag{1.4}$$

$$(q^2;q^2)_\infty\ \rho(q) = \sum_{n=0}^{\infty} \frac{(-1)^n (1-q^{4n+2}) q^{3n(n+1)}}{1+q^{2n+1}+q^{4n+2}}. \tag{1.5}$$

The following generalized Lambert series for 8th order mock theta functions are obtained by B. Gordon and R. J. McIntosh[9].

$$U_0(q) = 2\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(2n+1)}}{1+q^{4n}}, \tag{1.6}$$

$$U_1(q) = \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{(n+1)(2n+1)}}{1+q^{4n+2}}, \tag{1.7}$$

$$V_1(q) = \frac{(-q^4;q^4)_\infty}{(q^4;q^4)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{(2n+1)^2}}{1-q^{4n+1}}. \tag{1.8}$$

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We close this section by preparation of two identities as a special case of denominator identity for affine Lie superalgebras. We quote the denominator identity for affine Lie superalgebras which was discovered by V. G. Kac and M. Wakimoto [10], [15]. The identity is written in the following form

$$\frac{\prod_{\alpha\in\Delta^+_{\text{even}}}(1-e^{-\alpha})^{\text{mult}(\alpha)}}{\prod_{\alpha\in\Delta^+_{\text{odd}}}(1+e^{-\alpha})^{\text{mult}(\alpha)}} = e^{-\rho}\sum_{w\in W}\varepsilon(w)\ w\left(\frac{e^{\rho}}{\prod_{i=1}^{k}(1+e^{-\beta_i})}\right).$$

In this formula, Δ^+_{even} (resp. Δ^+_{odd}) is the set of all even (resp. odd) and positive roots of the affine Lie superalgebra $\widehat{\mathfrak{g}}$ and $\text{mult}(\alpha)$ is the dimension of the root space $\widehat{\mathfrak{g}}_\alpha$ and W is the Weyl group of $\widehat{\mathfrak{g}}$ and $\varepsilon(w)$ is the signature of $w \in W$ and ρ is the Weyl vector of $\widehat{\mathfrak{g}}$ and $\{\beta_1, \cdots, \beta_k\}$ is a maximal set of simple odd roots satisfying the inner product $(\beta_i|\beta_j) = 0$ for all $i, j = 1, \cdots, k$ (see [10], [11], [14], [15] for complete explanation and details).
In [15] , we can find that the denominator identity for $\widehat{A}(1,0)$ is

$$e^{\rho}R = \sum_{w\in W}\varepsilon(w)w\left(\frac{e^{\rho}}{1+e^{-\alpha_2}}\right),$$

where

$$R = \prod_{n=1}^{\infty}\frac{(1-e^{-n\delta})^2(1-e^{-(n-1)\delta-\alpha_1})(1-e^{-n\delta+\alpha_1})}{(1+e^{-(n-1)\delta-\alpha_2})(1+e^{-n\delta+\alpha_2})(1+e^{-(n-1)\delta-\alpha_1-\alpha_2})(1+e^{-n\delta+\alpha_1+\alpha_2})},$$

α_1 is an even simple root, α_2 is an odd simple root and δ is a primitive imaginary root. The identity is rewritten as follows:

$$\begin{aligned}(1.9)\quad & e^{\rho}\prod_{n=1}^{\infty}\frac{(1-e^{-n\delta})^2(1-e^{-(n-1)\delta-\alpha_1})}{(1+e^{-(n-1)\delta-\alpha_2})(1+e^{-n\delta+\alpha_2})} \\ & \quad\times\frac{(1-e^{-n\delta+\alpha_1})}{(1+e^{-(n-1)\delta-\alpha_1-\alpha_2})(1+e^{-n\delta+\alpha_1+\alpha_2})} \\ & = e^{\rho}\left(\sum_{n=-\infty}^{\infty}\frac{e^{-\delta n(n+1)}e^{n\alpha_1}}{1+e^{-\alpha_2}e^{-n\delta}} - \sum_{n=-\infty}^{\infty}\frac{e^{-\delta n(n+1)}e^{-\alpha_1(-n+1)}}{1+e^{-\alpha_1}e^{-\alpha_2}e^{-n\delta}}\right).\end{aligned}$$

By putting $q := e^{-\delta}$, $x := e^{-\alpha_1}$ and $y := e^{-\alpha_2}$, (1.9) is calculated as follows:

$$\begin{aligned}(1.10)\qquad & \sum_{n=-\infty}^{\infty}\frac{x^{-n}q^{n(n+1)}}{1+yq^n} - \sum_{n=-\infty}^{\infty}\frac{x^{n+1}q^{n(n+1)}}{1+xyq^n} \\ & = \frac{(q,q,x,q/x)_\infty}{(-y,-q/y,-xy,-q/xy)_\infty}.\end{aligned}$$

By replacing q by q^2 and substituting $x = q$ and $y = z$ in (1.10), we have

$$A(z;q) := \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(n+1)/2}}{1+zq^n} = \frac{(q,q)_\infty}{(-z,-z^{-1}q)_\infty}, \tag{1.11}$$

where $z \neq -q^n \quad (n \in \mathbf{Z})$.

In [15], we can also find that the denominator identity for $\widehat{B}(1,1)$ is

$$e^\rho R = \sum_{w\in W} \varepsilon(w) w \left(\frac{e^\rho}{1+e^{-\alpha_1}} \right),$$

where

$$\begin{aligned} R &= \prod_{n=1}^{\infty} \frac{(1-e^{-n\delta})^2(1-e^{-(n-1)\delta-\alpha_2})(1-e^{-n\delta+\alpha_2})}{(1+e^{-(n-1)\delta-\alpha_1})(1+e^{-n\delta+\alpha_1})(1+e^{-(n-1)\delta-\alpha_1-\alpha_2})} \\ &\quad \times \frac{(1-e^{-(n-1)\delta-2\alpha_1-2\alpha_2})(1-e^{-n\delta+2\alpha_1+2\alpha_2})}{(1+e^{-n\delta+(\alpha_1+\alpha_2)})(1+e^{-(n-1)\delta-\alpha_1-2\alpha_2})(1+e^{-n\delta+\alpha_1+2\alpha_2})}, \end{aligned}$$

α_1 is an odd simple root and α_2 is an even simple root. The identity is rewritten as follows:

$$\begin{aligned} & e^\rho \prod_{n=1}^{\infty} \frac{(1-e^{-n\delta})^2(1-e^{-(n-1)\delta-\alpha_2})(1-e^{-n\delta+\alpha_2})}{(1+e^{-(n-1)\delta-\alpha_1})(1+e^{-n\delta+\alpha_1})(1+e^{-(n-1)\delta-\alpha_1-\alpha_2})} \\ &\times \frac{(1-e^{-(n-1)\delta-2\alpha_1-2\alpha_2})(1-e^{-n\delta+2\alpha_1+2\alpha_2})}{(1+e^{-(n-1)\delta-\alpha_1-2\alpha_2})(1+e^{-n\delta+\alpha_1+2\alpha_2})(1+e^{-n\delta+(\alpha_1+\alpha_2)})} \\ &= e^\rho \left(\sum_{n=-\infty}^{\infty} \frac{e^{-\delta(\frac{1}{2}n^2+\frac{1}{2}n)} e^{-n(\alpha_1+\alpha_2)}}{1+e^{-\alpha_1}e^{-n\delta}} - \sum_{n=-\infty}^{\infty} \frac{e^{-\delta(\frac{1}{2}n^2+\frac{1}{2}n)} e^{-n(\alpha_1+\alpha_2)} e^{-\alpha_2}}{1+e^{-\alpha_1}e^{-2\alpha_2}e^{-n\delta}} \right). \end{aligned} \tag{1.12}$$

By putting $q := e^{-\delta}$, $x := e^{-\alpha_1}$ and $y := e^{-\alpha_2}$, (1.12) is calculated as follows:

$$\begin{aligned} & \sum_{n=-\infty}^{\infty} \frac{q^{\frac{1}{2}n^2+\frac{1}{2}n}(xy)^n}{1+xq^n} - \sum_{n=-\infty}^{\infty} \frac{q^{\frac{1}{2}n^2+\frac{1}{2}n}(xy)^{-n-1}}{1+x^{-1}y^{-2}q^n} \\ &= \frac{(q,q,y,q/y,x^2y^2,q/x^2y^2)_\infty}{(-x,-q/x,-xy,-q/xy,-xy^2,-q/xy^2)_\infty}. \end{aligned} \tag{1.13}$$

By replacing q by q^2 and substituting $x = zq^{1/2}$ and $y = z^{-1}q^{-1}$ in (1.13), we have

$$\begin{aligned} B(z;q) &:= \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(n+1)/4}}{1+zq^{n+\frac{1}{2}}} \\ &= \frac{(q^{\frac{1}{2}},q)_\infty (q^2,zq,z^{-1}q;q^2)_\infty}{(-zq^{\frac{1}{2}},-z^{-1}q^{\frac{1}{2}})_\infty}, \end{aligned} \tag{1.14}$$

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where $z \neq -q^{n+\frac{1}{2}} \quad (n \in \mathbf{Z})$.

2. The proof of Theorem 1.1

Proof. We need the following two identities.

$$(2.1) \qquad (q^2;q^2)_\infty \left(\omega(q)+\omega(-q)\right) = 2\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{3n(n+1)}}{1-q^{4n+2}},$$

$$(2.2) \qquad (q^2;q^2)_\infty \left(\rho(q)+\rho(-q)\right) = 2\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{3n(n+1)}}{1+q^{4n+2}+q^{8n+4}}.$$

The identity (2.1) follows from (1.4) :

$$\begin{aligned}
(q^2;q^2)_\infty \left(\omega(q)+\omega(-q)\right) &= \sum_{n=0}^{\infty}(-1)^n q^{3n(n+1)} \left(\frac{1+q^{2n+1}}{1-q^{2n+1}} + \frac{1-q^{2n+1}}{1+q^{2n+1}}\right) \\
&= \sum_{n=0}^{\infty}(-1)^n q^{3n(n+1)} \left(\frac{2(1+q^{4n+2})}{1-q^{4n+2}}\right) \\
&= 2\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{3n(n+1)}}{1-q^{4n+2}}.
\end{aligned}$$

Similarly, the identity (2.2) follows from (1.5). Two identities (2.1) and (2.2) are rewritten as follows:

$$\begin{aligned}
&(q^2;q^2)_\infty \left(\frac{\omega(q)+\omega(-q)}{2}\right) \\
&= \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{3n(n+1)}}{1-q^{4n+2}} \times \frac{1+q^{4n+2}+q^{8n+4}}{1+q^{4n+2}+q^{8n+4}} \\
&= \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{3n(n+1)}}{1-q^{12n+6}} + \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{3n(n+1)+4n+2}}{1-q^{12n+6}} \\
&\qquad + \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{3n(n+1)+8n+4}}{1-q^{12n+6}} \\
&= \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{3n(n+1)}}{1-q^{12n+6}} + \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{3n(n+1)+4n+2}}{1-q^{12n+6}} \\
&\qquad + \sum_{n=-\infty}^{\infty} \frac{(-1)^{-n} q^{3(-n)(-n+1)+8(-n)+4}}{1-q^{12(-n)+6}}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{3n(n+1)}}{1-q^{12n+6}} + \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{3n(n+1)+4n+2}}{1-q^{12n+6}} \\
&\qquad + \sum_{n=-\infty}^{\infty} \frac{(-1)^{-n} q^{3(-n)(-n+1)+8(-n)+4}}{-q^{-12n+6}(1-q^{12n-6})} \\
&= \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{3n(n+1)}}{1-q^{12n+6}} + \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{3n(n+1)+4n+2}}{1-q^{12n+6}} \\
&\qquad - \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{3n(n-1)+4n-2}}{1-q^{12n-6}} \\
&= \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{3n(n+1)}}{1-q^{12n+6}} + \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{3n(n+1)+4n+2}}{1-q^{12n+6}} \\
&\qquad - \sum_{n=-\infty}^{\infty} \frac{(-1)^{n+1} q^{3(n+1)n+4(n+1)-2}}{1-q^{12(n+1)-6}} \\
&= \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{3n(n+1)}}{1-q^{12n+6}} + 2\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{3n(n+1)+4n+2}}{1-q^{12n+6}}
\end{aligned}
$$

and

$$
\begin{aligned}
&(q^2;q^2)_\infty\,(\rho(q)+\rho(-q)) \\
&\quad = 2\left(\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{3n(n+1)}}{1+q^{4n+2}+q^{8n+4}} \times \frac{1-q^{4n+2}}{1-q^{4n+2}}\right) \\
&\quad = 2\left(\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{3n(n+1)}}{1-q^{12n+6}} - \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{3n(n+1)+4n+2}}{1-q^{12n+6}}\right),
\end{aligned}
$$

respectively. From these identities, the left hand side of Theorem 1.1 simplifies to

$$
\begin{aligned}
(2.3)\qquad &(q^2;q^2)_\infty\left((\rho(q)+\rho(-q)) + \left(\frac{\omega(q)+\omega(-q)}{2}\right)\right) \\
&= 3\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{3n(n+1)}}{1-q^{12n+6}} \\
&= 3B(-1;q^{12}).
\end{aligned}
$$

Replacing q by q^{12} and substituting $z=-1$ in (1.14), we have

$$
(2.4)\qquad B(-1;q^{12}) = \frac{(q^{12};q^{12})_\infty(-q^{12},\ -q^{12},\ q^{24};q^{24})_\infty}{(q^6;q^{12})_\infty}.
$$

By (2.3) and (2.4), the proof completes. □

This proof implies that the relation in Theorem 1.1 equals to the special case of the denominator identity for affine Lie superalgebra in the case $\widehat{B}(1,1)$.

3. Some identities

We will prove some identities relating mock theta functions by using the denominator identity. First we see that the identity (1.1):

$$(q)_\infty \left(4\chi(q) - f(q)\right) = 3\frac{(q^3;q^3)_\infty^2}{(-q^3;q^3)_\infty^2}$$

is derived from the denominator identity for $\widehat{A}(1,0)$. From (1.2) and (1.3), we have

$$(q)_\infty \left(4\chi(q) - f(q)\right) = 6 \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{3n(n+1)/2}}{1+q^{3n}} = 6A(1;q^3). \tag{3.1}$$

Replacing q by q^3 and substituting $z = 1$ in (1.11), we have

$$A(1;q^3) = \frac{(q^3;q^3)_\infty^2}{2\,(-q^3;q^3)_\infty^2}. \tag{3.2}$$

Hence, from (3.1) and (3.2), we can see that the identity (1.1) is a special case of the denominator identity for $\widehat{A}(1,0)$.
Next, we prove

$$(q^2;q^2)_\infty \left(\rho(q) + \frac{1}{2}\,\omega(q)\right) = \frac{3}{2}\,\frac{(q^6;q^6)_\infty^2}{(q^3;q^6)_\infty^2}, \tag{3.3}$$

which is obtained by G. N. Watson in [16]. From (1.4) and (1.5), we have

$$(q^2;q^2)_\infty \left(\rho(q) + \frac{1}{2}\omega(q)\right) = \frac{3}{2} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{3n(n+1)}}{1-q^{6n+3}} = \frac{3}{2}A(-q^3;q^6).$$

Replacing q by q^6 and substituting $z = -q^3$ in (1.11) which is type $\widehat{A}(1,0)$, we have

$$A(-q^3;q^6) = \frac{(q^6;q^6)_\infty^2}{(q^3;q^6)_\infty^2}.$$

Hence we can prove (3.3).
Finally, we prove two identities relating mock theta functions of order 8 in [9]:

$$U_0(q) + 2U_1(q) \quad = \quad (-q;q^2)_\infty^3 (q^2;q^2)_\infty (q^2;q^4)_\infty, \tag{3.4}$$

$$(3.5)\qquad V_1(q) - V_1(-q) \;=\; 2q(-q^2;q^2)_\infty(-q^4;q^4)^2_\infty(q^8;q^8)_\infty.$$

From (1.6) and (1.7), we have

$$(3.6)\qquad U_0(-q) + 2U_1(-q) \;=\; 2\frac{(q;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{\frac{n(n+1)}{2}}}{1+q^{2n}}$$
$$\;=\; 2\frac{(q;q^2)_\infty}{(q^2;q^2)_\infty}B(q^{-1};q^2).$$

Replacing q by q^2 and substituting $z = q^{-1}$ in (1.14), we have

$$(3.7)\qquad B(q^{-1};q^2) = \frac{1}{2}\frac{(q)^2_\infty}{(-q^2;q^2)_\infty}.$$

From (3.6) and (3.7), we prove (3.4) by replacing q by $-q$. Similarly, from (1.8), we have

$$V_1(q) - V_1(-q) \;=\; 2q\frac{(-q^4;q^4)_\infty}{(q^4;q^4)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{4n(n+1)}}{1-q^{8n+2}}$$
$$\;=\; 2q\frac{(-q^4;q^4)_\infty}{(q^4;q^4)_\infty}A(-q^2;q^8).$$

Replacing q by q^8 and substituting $z = -q^2$ in (1.11), we have

$$A(-q^2;q^8) = \frac{(q^8;q^8)^2_\infty}{(q^2;q^8)_\infty(q^6;q^8)_\infty} = \frac{(q^8;q^8)^2_\infty}{(q^2;q^4)_\infty}.$$

Hence, we prove (3.5).

Remark. The denominator identities for affine Lie superalgebras in the case $\widehat{A}(1,0)$ and $\widehat{B}(1,1)$ are related to Ramanujan's summation formula ${}_1\psi_1$ series and Bailey's summation formula of a very-well-poised-balanced ${}_6\psi_6$ series, respectively.

4. Appendix

Here is an additional remark. We shall show that the identity in Theorem 1.1 can be seen as a relation between theta constants. From (3.3), the identity in Theorem 1.1 becomes

$$(4.1)\qquad j(-q^3,q^{12})^2 + j(q^3,q^{12})^2 = 2\,j(-q^6,q^{24})\,j(-q^{12},q^{24}).$$

Now, let v be a complex number and τ be a complex number whose imaginary part is positive. Theta functions are defined by

$$\vartheta_3(v|\tau) = \vartheta_{00}(v,\tau) \;:=\; \sum_{n=-\infty}^{\infty}\mathbf{e}\left(\frac{1}{2}n^2\tau + nv\right),$$

$$\vartheta_4(v|\tau) = \vartheta_{01}(v,\tau) \quad := \quad \sum_{n=-\infty}^{\infty} \mathbf{e}\left(\frac{1}{2}n^2\tau + n\left(v+\frac{1}{2}\right)\right),$$

where $\mathbf{e}(x) := e^{2\pi i x}$ [12]. Putting $z = e^{v\pi i}$ and $q = e^{\tau\pi i}$, theta functions can be rewritten as product formulas:

$$\vartheta_{00}(v,\tau) = j(-z^2q, q^2), \quad \vartheta_{01}(v,\tau) = j(z^2q,q^2).$$

In the following relation[7, §13.23.(Transformations of the second order)]

$$\vartheta_{00}(v,\tau)^2 + \vartheta_{01}(v,\tau)^2 = 2\ \vartheta_{00}(0,2\tau)\ \vartheta_{00}(2v,2\tau), \tag{4.2}$$

replacing τ by 6τ and v by $\frac{3}{2}\tau$ and using $j(x,q) = j(q/x,q)$ yields (4.1). Hence, we can see that the identity in Theorem 1.1 is also the special case of (4.2).

5. Conclusion

In [15, p195], M. Wakimoto states that "The denominator identities for the simplest affine Lie superalgebras are Ramanujan's mock theta functions. In this sense, denominator identities of affine Lie superalgebras provide a general class of mock theta functions." However, the specific instance for 3rd or 8th order mock theta functions is not given there. We have found that some identities satisfied by mock theta functions are special cases of the denominator identity. It is plausible that these connections will assist in giving the true meaning of mock theta functions.

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Mock theta functions were introduced by Ramanujan in 1920. They have become a vivid area of research, and they continue to play important roles in different parts of mathematics and physics. In this paper, we extend the concept of holomorphic projection, which allows us to prove identities for the Fourier series coefficients of Ramanujanâ€™s mock theta functions.Â Mock theta functions have a long history but recent work establishes surprising connections with different areas of mathematics and physics. For example, they impact the theory of Donaldson invariants of that are related to gauge theory (for example, refs.Â 14. Scott Ahlgren mentioned to us that it could also be derived from ref. 15, because is congruent modulo 2 to the generating function for the partition function. rem 1.1) provides four transformation formulas relating the new mock theta functions with Ramanujanâ€™s mock theta functions of the sixth order. Two further representations of the new mock theta functions are established.Â In Theorem 1.1 below, we offer four identities, which are similar in spirit to identities for sixth order mock theta functions stated by Ramanujan in his lost notebook [13] and proved by Andrews and D. Hickerson [3]. Most of this paper is devoted to proving these four identities. In fact, R.J. McIntosh [11] inde-pendently discovered these two mock theta functions and derived transformation formulas for them. There are no theorems in [11] in common with those proved in our paper. Throughout this article, we assume that |q| < 1 and use the notation. identify credible information, collecting, and analyzing credible information from a variety of local, national and international sources, including those in multiple languages. Education and society are closely_____. interrelated, interconnected.Â Unfortunately, many people who prepare for intercultural encounters might only _ which is not enough. gather information about the customs of the other country,learn a bit of the language. Each child is unique and there are no two _____similar people. Product identities in two variables x, q expand infinite products as infinite sums, which are linear combinations of theta functions; famous examples include Jacobi's triple product identity, Watson's quintuple identity, and Hirschhorn's septuple identity. We view these series expansions as representations in canonical bases of certain vector spaces of quasiperiodic meromorphic functions (related to sections of line and vector bundles), and find new identities for two nonuple products, an undecuple product, and several two-variable Rogers-Ramanujan type sums.Â You are currently offline. Some features of the site may not work correctly. Corpus ID: 221655290. Several new product identities in relation to two-variable Rogers-Ramanujan type sums and mock theta functions. A method is developed for obtaining Ramanujanâ€™s mock theta functions from ordinary theta functions by performing certain operations on their q-series expansions. The method is then used to construct several new mock theta functions, including the first ones of eighth order. Summation and transformation formulae for basic hypergeometric series are used to prove that the new functions actually have the mock theta property. The modular transformation formulae for these functions are obtained. View. Show abstract. Generalized Lambert series identities. Article. Nov 2005. Song Heng Chan.Â Some identities relating mock theta functions which are derived from denominator identity. January 2009.