On the Results of a 14-Year Effort to Generalize Gödel's Second Incompleteness Theorem and Explore Its Partial Exceptions

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The Second Incompleteness Theorem states that axiom systems of sufficient strength are unable to verify their own consistency. Part of the reason that Gödel's theorem has fascinated logicians is that it almost defies common sense. This is because when human beings cogitate, they implicitly presume that thinking is a useful process. However, this tacit assumption would initially appear to presume that logic is consistent (since an inconsistent logic formalism is known to prove all true and false theorems — thereby rendering logical deduction quite useless). Thus, the Second Incompleteness Theorem seems to suggest that it is almost impossible — and certainly very awkward — to formalize the common-sense assumption that thinking is useful (because sufficiently strong axiom systems are unable to recognize their own consistency.)

The preceding mystery about the nature of incompleteness lies at the heart of the scholarly community's fascination with Gödel's historic discovery [3]. Several of Gödel's biographers [4, 21] have also noted that Gödel explicitly hedged in one of the closing paragraphs in his paper [5] about whether or not the Second Incompleteness Theorem should be interpreted as meaning all efforts for an axiom system to verify its own consistency by finite means are baseless. For instance, page 58 of Yourgrau's biography [21] of Gödel points out that Von Neumann viewed the Second Incompleteness Theorem as having a much broader range of applications than Gödel for several years after the publication of Gödel's seminal 1931 paper.

What we have sought to do during the last 14 years was to simultaneously explore paradigms where an unusual axiom system can formalize at least a partial conception

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of its own internal consistency, as well as to develop new generalizations of the Second Incompleteness Theorem. In most of our recent articles, we have deliberately used the phrase “Boundary Case Exception” to describe the partial respects in which an axiom system can evade the incompleteness phenomena. This is because the Second Incompleteness Theorem is of course too powerful for a full-fledged exception to it to ever arise. (Therefore the cautious phrase “boundary case exception” is quite appropriate.)

There are two reasons as to why such partial exceptions to the Second Incompleteness Theorem are of scholarly interest. The first is that the academic community’s knowledge of the meaning of the Second Incompleteness Theorem is significantly sharpened by exploring the broadest generalizations of the incompleteness phenomena, as well as by investigating the partial circumstances where the Second Incompleteness Theorem fails to apply. Especially when Gödel’s Incompleteness Theorem is often considered the paramount discovery of 20th century mathematics, it beckons the scholarly community to classify its maximum generalization and partial limitations.

The second point is that while unconventional logics *will certainly lack some of the mathematical features that one would prefer a logic formalism to ideally encompass*, these unconventional formalisms still do possess certain partial virtues. This is because it is undeniable that human beings choose to think because they instinctively believe that cogitation is a useful process. Unconventional logics can offer a means to at least partially formalize how it is that human beings retain some type of partial knowledge of their own consistency.

Thus in order to achieve these two goals, we have published approximately a dozen articles over the last 14 years, including three papers in the *Journal of Symbolic Logic* and a fourth paper in the *Annals of Pure and Applied Logic* [13, 14, 17, 19]. The remainder of this extended abstract will summarize several of these results. We shall make no claim that the unconventional logics, employed by our proposed self justifying axiom systems, are a full answer to the question of: *“Why do human beings choose to think?”* Indeed, there is likely to be no full answer to this question because most philosophical questions are never fully resolved.

Notation Conventions: Let $A(x,y,z)$ and $M(x,y,z)$ denote two 3-way predicates indicating $x+y=z$ and $x*y=z$. Also, our published papers have used the term " Π_1^- " sentence to refer to a notion identical to classic arithmetic's Π_1 sentences except that addition and multiplication are treated as 3-way predicates instead of as function symbols. Another notation convention is that a formal system α will be said to **recognize** addition and multiplication as **Total Functions** iff it contains the axioms: "$\forall x \; \forall y \; \exists z \;\; A(x,y,z)$" and "$\forall x \; \forall y \; \exists z \;\; M(x,y,z)$"

All our efforts to construct axiom systems that partially evade the Second Incompleteness Theorem have involved studying formal systems that have dropped the formal axiom declaring that integer-multiplication is a total function. These systems can treat multiplication as a 3-way predicate, and prove all the Π_1^- theorems of Peano Arithmetic. *We do not wish to minimize* the difficulties inherent in studying formal systems that drop the axiom that multiplication is a total function. However, our thesis is that there are also some benefits inherent from such an investigation because it offers at least a partial explanation to the questions of *"Why and How do human beings have a partial instinctive faith about the consistency and usefulness of their thought processes"*. For instance, we have recently published a paper [18] showing that our self-justifying axiom systems can recognize an infinitized analog of a computer's floating point instruction set as a total function. Thus while the dropping of the axiom declaring integer multiplication is a total function is unquestionably a real sacrifice, our formalisms are able unify an understanding of the capacities of floating point multiplication with at least some form of limited understanding of their own self consistency.

Underlying Intuitions: Given any ordered pair (α, D) specifying an axiom system α and a deduction method D, Kleene [6] noticed in 1938 that it was technically feasible to define a second axiom system $\alpha^D \supset \alpha$ that at least "formally" recognized its own consistency. Kleene's proposal was for α^D to essentially consist of all of α's axioms plus the following additional sentence, which is called Diagonal(α, D) :

> There is no proof (using deduction method D) of the "falsity sentence" $0=1$ from the union of the axiom system α with $this$ sentence "Diagonal(α, D)" (looking at itself).

Several of Kleene's students have recalled how he would typically explain in a logic class how the twin concepts of Diagonal(α, D) and α^D were well defined constructs and then challenge the students of his class to explain why α^D did not contradict Gödel's Second Incompleteness Theorem (since α^D does retain an ability to use its axiom Diagonal(α, D) to prove its own consistency).

The answer is that while Diagonal(α, D) and α^D are indisputably well defined constructs, the axiom system α^D is typically inconsistent despite the fact that its axiom system Diagonal(α, D) declares its own consistency. Indeed, a Gödel-like diagonalization will lead one to conclude that the very presence of the axiom Diagonal(α, D) within the formalism of α^D will typically cause the latter to become inconsistent and therefore useless.

One of our research objectives has been to identify circumstances where the initial axiom system α will be sufficiently weak so that the resulting system α^D can somehow manage to escape a Gödel-like diagonalization argument and be consistent. The challenge of finding (α, D) that can escape this paradigm is non-trivial. This is because a version of the Second Incompleteness Theorem, that is essentially due to the combined work of Pudlák and Solovay [7, 8], implies that it is impossible to construct any consistent axiom system that simultaneously recognizes successor as a total function, views addition and multiplication as 3-way plausibly non-total relations, and which retains a capacity to recognize its own Hilbert consistency.

Thus in light of these challenges, our research has sought to develop new generalizations of the Second Incompleteness Theorem as well as to classify its boundary-case exceptions — so that the exact meaning and generality of Gödel's historic theorem can be better understood.

If a researcher wishes to read only one of our eleven papers listed in the bibliography section, then we recommend that [17] be the article that is examined first. Below is a summary of our eleven articles, illustrating both their both positive and closely-matching negative opposing results:

A. The papers [10, 13] were our initial results appearing in conference and journal formats. Their most memorable result was to show that for any consistent axiom system A, it is possible to construct a system IS(A) that can simultaneously internally recognize its own cut-free consistency, the validity of all A's Π_1^- theorems and that integer addition is a total function.

B. The papers [12, 14] formally proved that the above partial exception to the Second Incompleteness Theorem fails to generalize when an axiom system recognizes integer multiplication as a total function. They also resolved a 20-year old Paris-Wilkie open question [9] by showing that the *standard and classical textbook axiomization* of IΣ_0 satisfies a version of Gödel's Second Incompleteness Theorem applying to cut-free deduction. (Prior to our work, [1, 2] had shown that I$\Sigma_0 + \Omega_1$ could not recognize its cut-free consistency. We strengthened the over-all formalism so that the cut-free version of the Second Incompleteness Theorem would apply to IΣ_0.)

C. Let us define Level$-k$ deduction to be a modified form of semantic tableaux that allows a Gentzen-style deductive cut rule to be applied to Π_k^- and Σ_k^- sentences. The article [17] showed that our boundary-case exceptions to the Second Incompleteness Theorem can be extended from classic cut-free deduction to Level-1 deduction. In contrast, [16] formalized a generalization of the Second Incompleteness Theorem that showed a similar result would not extend to Level-2.

D. The article [18] showed that self-justifying axiom systems can recognize floating point multiplication as a total function. (This adds a level of robustness to a formalism that can treat floating point multiplication in a much more flexible manner than integer multiplication.) Moreover, [18] observed that its partial exception to the Second Incompleteness Theorem will nicely hybridize with [17]'s result (summarized in Item C) so that for any initial axiom system A that is valid under the standard model of the natural numbers, one can construct a system $I(A)$ which can:
   1. recognize the validity of all A's Π_1^- and Σ_1^- theorems;
   2. recognize integer addition and all the floating point operations of real-number arithmetic (including floating point multiplication) as total functions; and
   3. recognize its own internal self consistency under the Level-1 deduction method (defined by Item C).

   The over-all theme of [18]'s discussion is *that while Gödel's Second Incompleteness Theorem certainly generalizes to Floating Point Arithmetic,* it nevertheless has the property that Numerical Analysis supports *much more robust partial exceptions* to the Second Incompleteness Theorem than Number Theory can do !

E. Consider a special sequence of “named constant symbols” $C_0, C_1, C_2, \ldots$ where $C_0 = 2$ and where

1. $C_{i+1} = C_i + C_i$ under the “Additive Naming Convention”, and
2. $C_{i+1} = C_i * C_i$ under the “Multiplicative Naming Convention”.

The article [19] considers a paradigm where none of addition, multiplication or successor are viewed as total functions. (The Pudlák-Solovay version of the Second Incompleteness Theorem for Hilbert Deduction is thus inapplicable.) Instead, the axiom systems of [19] formalize the infinite growth among integers by using either the additive or multiplicative “naming conventions”. The two main theorems of [19] are that:

1. It is possible to develop very strong boundary-case exceptions to the Second Incompleteness Theorem when the Additive Naming convention is present (involving a system recognizing its own Hilbert consistency and proving all Peano Arithmetic's Π_1^- theorems).
2. In contrast, the Second Incompleteness Theorem does hold for Hilbert deduction when the multiplicative naming convention is present.

F. Our paper [20] establishes the surprising result that an unconventional axiomatization of $\mathrm{I}\Sigma_0$ proves the same theorems as its conventional axiomatization, but the former (unlike the latter) is an anti-threshold for the Herbrandized version of the Second Incompleteness Theorem. (The intuitive reason that these two logically equivalent systems have opposite threshold properties is that these two systems are unable to prove their equivalence to each other — although they are equivalent).

G. The articles [11, 13] discuss an additional invariant that most of our self-justifying axiom systems will satisfy called the Tangibility Reflection Principle.

The general theme of our research is that the Second Incompleteness Theorem possesses philosophically interesting boundary-case exceptions, although full-fledged execptions to it certainly do not exist

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GA¶delâ€™s first incompleteness theorem, however, states that some arithmetical truths are not provable because that would require a formal system that incorporates methods going beyond the arithmetical system used to derive them. The Second and Final Incompleteness Theorem. For any consistent system F within which a certain amount of elementary arithmetic can be carried out, the consistency of F cannot be proved in F itself. This is an extension of the first incompleteness theorem and goes to show that a formal system which claims itself to be consistent cannot prove that it doesnâ€™t have any contradictions. In the case of the second theorem, F must contain a little bit more arithmetic than in the case of the first theorem. tizer-Prize-winning G6deL Escber, Bach 9 GOdel's incompleteness theorem, con- GOdel's incompleteness theorems.Â Steering arithmetic making use of a generalized GOdel proved that any b o o k o n the in- the b e g i n n i n g student clear of some version of mathematical induction, completeness theorem must b e incom- c o m m o n confusions, Franzdn explains k n o w n as transfinite induction. Why plete or inconsistent.Â theorem the physicist's dream of a TheolT of tificial Intelligence (AI) are d o o m e d to is a matter of inspiration rather than Everything is not only unattained, but failure, going on to conjecture that a implication": theoretically unattainable. GÃ¶delâ€™s two incompleteness theorems are among the most important results in modern logic, and have deep implications for various issues. They concern the limits of provability in formal axiomatic theories.Â There have been attempts to apply the results also in other areas of philosophy such as the philosophy of mind, but these attempted applications are more controversial. The present entry surveys the two incompleteness theorems and various issues surrounding them. (See also the entry on Kurt GÃ¶del for a discussion of the incompleteness theorems that contextualizes them within a broader discussion of his mathematical and philosophical work.) 1. Introduction. 1.1 Outline. The incompleteness theorems after 70 years. Annals of Pure and Applied Logic, Vol. 126, Issue. 1-3, p. 125.Â About the characterization of a fine line that separates generalizations and boundary-case exceptions for the Second Incompleteness Theorem under semantic tableau deduction. Journal of Logic and Computation, Vol. 31, Issue. 1, p. 375.