Problems about rationality of K3 and del Pezzo surfaces

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K3 surfaces

Let S be a surface. We say that S is a *K3 surface* if its canonical divisor K_S is trivial and $\dim H^1(S, \mathcal{O}_S) = 0$. Examples of K3 surfaces are smooth quartic surfaces in $\mathbb{P}^3$. Little is known about the arithmetic of K3 surfaces.

Question 1. *Is there a K3 surface S defined over $\mathbb{Q}$ or any number field such that its set of rational points is neither empty nor Zariski dense?*

In [3], Logan, McKinnon, and van Luijk prove the Zariski density of rational points for a large family of K3 surfaces. In [2], using similar techniques, we prove the following theorem.

Theorem 2. *Let c_1, c_2 be two nonzero rationals and S be the surface defined as*

$$S\colon x^4 - 4c_1^2 y^4 - c_2 z^4 - 4c_2 w^4 = 0.$$

Let $P = (x_0 : y_0 : z_0 : w_0)$ be a rational point on S with x_0 and y_0 both nonzero.
If $|2c_1|$ is a square in $\mathbb{Q}^\times$, then also assume that z_0, w_0 are not both zero. Then the set of rational points on the surface is Zariski dense.

Let S be a K3 surface defined over a number field k. We say that the rational points on S are *potentially dense* if there exists a finite extension k'/k such that the set $S(k')$ of k'-rational points is Zariski dense. In [1] Bogomolov and Tschinkel prove the following theorem.

Theorem 3. *Let S be a K3 surface defined over a number field k. Assume that S has a structure of an elliptic fibration or an infinite group of automorphisms. Then the rational points on S are potentially dense.*

Question 4. *Let S be a K3 surface defined over a number field k. Assume that S does not admit an elliptic fibration and its group of automorphisms is finite. Are the rational points of S potentially dense?*

Del Pezzo surfaces

We say that a variety X of dimension n over a field k is *unirational* if there exists a dominant rational map $\mathbb{P}^n \dashrightarrow X$, defined over K. A *del Pezzo surface* is a smooth, projective, geometrically integral variety X of which the anticanonical divisor $-K_X$ is ample. We define the *degree* of a del Pezzo surface X as the self intersection number of K_X, that is, $\deg X = K_X^2$.
In [4] it is proven that many del Pezzo surfaces defined over a field k having a rational point are unirational. Building on work by Manin (see [4]), C. Salgado, D. Testa, and A. Várilly-Alvarado prove that all del Pezzo surfaces of degree 2 over a finite field are unirational as well, except possibly for three isomorphism classes of surfaces (see [5]). Recently, I and Ronald van Luijk proved that these remaining three cases are also unirational, thus proving the following theorem.

Theorem 5. *Every del Pezzo surface of degree 2 over a finite field is unirational.*

Question 6. *What about del Pezzo surfaces of degree 2 over infinite fields and del Pezzo surfaces of degree 1 over any field?*

References

[1] F. A. Bogomolov and A. Tschinkel, *Density of rational points on elliptic K3 surfaces*, Asian J. Math. **4** (2000), no. 2, 351–368.

[2] D. Festi, *Density of rational points on a family of diagonal quartic surfaces*, Master thesis, Universiteit Leiden.

[3] A. Logan, D. McKinnon, and R. Van Luijk, *Density of rational points on diagonal quartic surfaces*, Algebra and Number Theory **4(1)** (2008), 1–20.

[4] Yu. I. Manin, *Cubic forms: algebra, geometry, arithmetic*, North-Holland publishing, 1986.

[5] Cecília Salgado, Damiano Testa, and Anthony Várilly-Alvarado, *On the unirationality of del Pezzo surfaces of degree two*, preprint (2013).

K3 surfaces (like abelian surfaces) are between Fano (del Pezzo) surfaces, with Ï‰Xâˆ'1 ample, and surfaces of general type, with Ï‰X ample. Arithmetic. Theorem (Segre, Manin, KollÂ´ar). Let X /k be a del Pezzo surface with Ï‰Xâˆ'1 very ample. Then X is unirational if and only if X (k) = âˆ…. Conjecture (Colliot-ThÂ´el`ene). Let X be a del Pezzo surface over a global eld k. If X (k) = âˆ…, then X (k) is Zariski dense in X . Conjecture (Bombieriâ€“Lang). Let X be a surface of general type over a nitely generated eld k. Then X (k) is not Zariski dense.Â Open problem 2. Is there a K3 surface X over a number eld k with X (k) neither empty nor Zariski dense? Question. Is there a K3 surface X over a number eld with Ï(X) = 1? The Severi Problem for Rational Curves on del Pezzo Surfaces. by Damiano Testa. Laurea, Universita` degli Studi di Roma â€œLa Sapienza,â€ July 2001. Submitted to the Department of Mathematics in partial fulllment of the requirements for the degree of. Doctor of Philosophy at the. MASSACHUSETTS INSTITUTE OF TECHNOLOGY June 2005. c Damiano Testa, MMV.Â 1. the valences of the vertices Ca and Cb in the dual graph of C are at most 3, and. 2. the map fâˆ— : TCa,p + TCb,p âˆ'â†' TS,f(p) is surjective. Then. H1(C, f âˆ—TS) âˆ¼= H1(CËœ, fËœâˆ—TS). Proof. Consider the sequence on C. 0 âˆ'â†' f âˆ—TS âˆ'â†' fËœâˆ—TS âˆ'â†'Îµ TS,f(p) âˆ'â†' 0. Abstract We study singular del Pezzo surfaces that are quasi-smooth and well-formed weighted hypersurfaces. We give an algorithm to classify all of them. Keywords: Fano variety; del Pezzo surface; KÂ¨ahlerâ€“Einstein metric. 2010 Mathematics subject classication: Primary 14J45 Secondary 32Q20; 14J26; 14Q10.Â The classication problem for Fano manifolds is closely related to the problem of the existence of KÂ¨ahlerâ€“Einstein metrics on them (see [27]). It has been conjectured by Yau, Tian and Donaldson that a Fano manifold admits a KaÂ¨hlerâ€“Einstein metric if and only if it is K-polystable. One direction of this conjecture, the K-polystability of KaÂ¨hlerâ€“ Einstein Fano manifolds, follows from the works of Tian, Donaldson, Stoppa and Berman (see [4, 14, 28, 31]). Modulus Space Hilbert Scheme Hodge Structure Pezzo Surface Monodromy Group. These keywords were added by machine and not by the authors. This process is experimental and the keywords may be updated as the learning algorithm improves.Â I am grateful to CIME and CIRM for organizing the school â€˜Rationality problems in algebraic geometryâ€™ where this material was to be presented, and the organizers Rita Pardini and Gian Pietro Pirola for supporting this survey despite my inability to lecture at the school. References. [ABBVA14]. A. Auel, M. Bernardara, M. Bolognesi, A. VÃ¡rilly-Alvarado, Cubic fourfolds containing a plane and a quintic del Pezzo surface. Algebr. Geom.